Problem Set #9

Suggested Solutions

1. (JR #9.5)

One could actually repeat verbatim the argument given in JR (pp. 379) regarding the optimal strategy in a second-price auction. The catchphrase there is "…*regardless of the bids submitted by the other bidders*..." as it clearly indicates that the argument would still be valid even if the joint distribution of the bidders' valuations exhibited correlation.

However, I will attempt here to give a rather more detailed presentation of their argument.

Let $r_i = \max_{j \neq i} b_j$ the maximum bid submitted by the other players.

Suppose first that player *i* is considering bidding $b_i > v_i$. We have the following possibilities:

- If $r_i \geq b_i > v_i$, bidder *i* does not win the object (or he does so in a tie with other players –i.e. which is a probability-measure zero event)[1]. In any case, his expected payoff is zero and it would be *exactly the same* had he bid v_i.
- If $b_i > r_i > v_i$, player *i* wins the object but gets a payoff of $v_i - r_i < 0$; he would be strictly better off by bidding v_i and not getting the object (i.e. stay with a payoff of zero).
- If $b_i > v_i > r_i$, player *i* wins the object and gets a payoff of $v_i - r_i \geq 0$. However, his payoff would be *exactly the same* if he were to bid v_i.

The reasoning is similar for the case $b_i < v_i$:

- When $r_i \leq b_i$ or $r_i \geq v_i$, the bidder's expected payoff would be *unchanged* if he were to bid v_i instead of b_i. However, if $b_i < r_i < v_i$, the bidder forgoes a positive payoff by underbidding. He currently loses the object (and gets zero payoff) whereas, had he bid v_i, he would have won it and obtained a payoff of $v_i - r_i > 0$.

This argument establishes that bidding one's valuation is a weakly dominant strategy[2]. By the very definition of weak dominance, this means that bidding one's valuation is

[1] In any auction, there is the question of how to resolve the ties –i.e. what happens when several bidders submit the highest bid. We usually take the auctioned object to be, in such a case, randomly allocated amongst them. The exact probability determining the allocation is irrelevant when the players' valuations are drawn from a continuous joint distribution and the bidding strategies are, in equilibrium, strictly increasing in the players' valuations. This is because, in such a setting, reaching a tie would mean that several players are submitting exactly the same bid. Given that bids are strictly monotone in players' valuations, this means that several bidders' valuations have been drawn to be the same point value. With a continuous joint distribution for the players' valuations, this is a zero-probability event. Consequently, in the case of ties, the winner of the object has the same zero expected payoff as the other players who will not get the object (since we expects to get it with zero probability).
In other words, when the players' valuations are drawn from a continuous joint distribution and the bidding strategies are, in equilibrium, strictly increasing in the players' valuations, how ties are resolved does not really matter because ties themselves are events that do not matter in expectation.

[2] The underlined preceding text shows clearly why it is only *weakly* dominant.

weakly optimal **no matter** what the bidding strategies of the other players are. It is, therefore, irrelevant how the other players' strategies are related to their own valuations, or to your own strategy (by, for example, their own valuations being correlated to your valuation). In other words, bidding one's valuation is weakly optimal **irrespectively** of any correlation in the joint distribution of the players' valuations (and also of whether the bidders have information about one another's valuations etc.)

2. (JR #9.8)

(a) Let us label the bidders by $i, j \in \{1,2\}$. Bidder *i* has valuation v_i for the good and the two bidders' valuations are identically and independently uniformly distributed on $[0,1]$ ($v_i \overset{i.i.d}{\sim} U[0,1]$ for $i, j \in \{1,2\}$).

Note that we will assume that bids are constrained to be non-negative.

In order to formulate this problem as a static Bayesian game of incomplete information, we must identify the action spaces, the type spaces, the beliefs and the payoff-functions. Player *i*'s action is to submit a (non-negative) bid b_i, her type is her valuation v_i, her action space is $A_i = [0,\infty)$ and her type space is $T_i = [0,1]$. Note that, because the valuations are independent, player *i* believes that v_j is uniformly distributed according to $U[0,1]$, *no matter* what the realization v_i of her own valuation is.

Finally, her payoff function is

$$u_i(b_1,b_2;v_1,v_2) = \begin{cases} v_i - b_j & i \text{ submits winning bid} \\ -b_i & i \text{ submits losing bid} \end{cases}$$

To derive a Bayesian Nash equilibrium (BNE) for this game, we begin by constructing the players' strategy spaces. In a static Bayesian game, a **strategy** is a function *from types to actions*. Hence, a strategy for player *i* is a function $b_i(v_i)$ specifying the bid that each of player *i*'s types (i.e. valuations) is supposed to submit. In a Bayesian Nash equilibrium, player *i*'s strategy $b_i(v_i)$ must be a best response to player *j*'s strategy $b_j(v_j)$ and vice versa. Formally, the pair of strategies $\left(b_i(v_i), b_j(v_j)\right)$ constitutes a BNE, if for each $v_i \in [0,1]$, $b_i(v_i)$ solves:

$$\max_{b_i} \ (v_i - b_j)\Pr\left[b_i > b_j(v_j)\right] + (-b_i)\Pr\left[b_i < b_j(v_j)\right] \Leftrightarrow$$

$$\max_{b_i} \ (v_i - b_j)\Pr\left[b_i > b_j(v_j)\right] + (-b_i)\left(1 - \Pr\left[b_i > b_j(v_j)\right]\right) \Leftrightarrow$$

$$\max_{b_i} \ -b_i + \left(v_i + b_i - b_j(v_j)\right)\Pr\left[b_i > b_j(v_j)\right]$$

To solve for a BNE, suppose that player *j* adopts the strategy $b(.)$ and assume that $b(.)$ is strictly increasing and differentiable. Then for a given realization of v_i, player *i*'s optimal bid solves:

$$\max_{b_i} \; -b_i + \left(v_i + b_i - b\left(v_j\right)\right)\Pr\left[b_i > b\left(v_j\right)\right] \qquad \text{(I)}$$

Let $b^{-1}\left(b_j\right) = b^{-1}\left(b\left(v_j\right)\right) = v_j$ the valuation that player *j* must have in order to be bidding b_j. Since $v_j \sim U[0,1]$ we have[3]:

$$\begin{aligned}
&-b_i + \left(v_i + b_i - b\left(v_j\right)\right)\Pr\left[b_i > b\left(v_j\right)\right] \\
&= -b_i + \left(v_i + b_i - b\left(v_j\right)\right)\Pr\left[b^{-1}\left(b_i\right) > b^{-1}\left(b\left(v_j\right)\right)\right] \\
&= -b_i + \left(v_i + b_i - b\left(v_j\right)\right)\Pr\left[b^{-1}\left(b_i\right) > v_j\right] \\
&= -b_i + \left(v_i + b_i - b\left(v_j\right)\right)\frac{b^{-1}\left(b_i\right)}{(1-0)}
\end{aligned}$$

Thus, the first order condition for player *i*'s optimization problem is:

$$-1 + \left(v_i + b_i - b\left(v_j\right)\right)\frac{db^{-1}\left(b_i\right)}{db_i} + b^{-1}\left(b_i\right) = 0 \qquad \text{(II)}$$

The first order condition (II) is an implicit equation for bidder *i*'s best response to the strategy $b(.)$ played by bidder *j* -given that bidder *i*'s valuation has been realized as v_i. If we are looking for a symmetric BNE, we require that both bidders play the same strategy in equilibrium. Since, therefore, bidder *j* plays the strategy $b(.)$, this must be also played by bidder *i*, in equilibrium. Hence, we require that $b(.)$ is player *i*'s best response to $b(.)$ by player *j*. In other words, $b\left(v_i\right)$ must satisfy the first order condition (II): that is, for each of bidder *i*'s positive valuations, she does not wish to deviate from bidding according to the schedule $b(.)$, given that player *j* bids according to the same schedule. To impose this requirement, we substitute $b_i = b\left(v_i\right)$ into (II):

$$-1 + \left(v_i + b\left(v_i\right) - b\left(v_j\right)\right)\frac{db^{-1}\left(b\left(v_i\right)\right)}{db_i} + b^{-1}\left(b\left(v_i\right)\right) = 0 \Leftrightarrow$$

$$\left(v_i + b\left(v_i\right) - b\left(v_j\right)\right)\frac{dv_i}{db_i} + v_i = 1 \Leftrightarrow$$

$$\left(v_i + b\left(v_i\right) - b\left(v_j\right)\right)\left(\frac{db_i}{dv_i}\right)^{-1} + v_i = 1$$

[3] It is in the second line of the following equation that we are using the assumption that the each player's bidding strategy is strictly increasing in her own valuation. One should always verify that the suggested BNE strategies obtained at the end are indeed strictly increasing in the players' own valuations.

Our last equation must be viewed as a first-order differential equation that the function $b(.)$ must satisfy. Clearly, however, if this is to be satisfy for *any* values of v_i, v_j, it should be so for $v_i = v_j = v$. We now have:

$$\left(v + b(v) - b(v)\right)\left(\frac{db}{dv}\right)^{-1} + v = 1 \Leftrightarrow$$

$$\frac{v}{b'(v)} + v = 1 \Leftrightarrow$$

$$b'(v) = \frac{v}{1-v} \Leftrightarrow$$

$$b(v) = \int \frac{v}{1-v} dv = \int \left(-1 + \frac{1}{1-v}\right) dv = -v - \ln(1-v) + c$$

where *c* is the constant of integration.

To eliminate *c* we need a boundary condition. Fortunately, simple economic reasoning provides us with one: no player should bid more than his/her valuation. Thus, we require $b(v_i) \le v_i \quad \forall v_i \in [0,1]$. In particular, we require $b(0) \le 0$. Since bids are constrained to be non-negative, this implies that $b(0) = 0$.

Hence, $c = 0$ and our proposed BNE solution is that each bidder submits a bid according to the schedule[4]: $b(v_i) = -v_i - \ln(1 - v_i)$

(b) Consider now a first-price, all-pay auction[5].

Player *i*'s payoff function now is:

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i & i \text{ submits winning bid} \\ -b_i & i \text{ submits losing bid} \end{cases}$$

The pair of strategies $\left(b_i(v_i), b_j(v_j)\right)$ constitutes a BNE here, if for each $v_i \in [0,1]$, $b_i(v_i)$ solves:

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[4] Note that: $b'(v) = -1 + \frac{1}{1-v} = \frac{v}{1-v} > 0 \quad \forall v \in (0,1]$ whereas $b'(0) = 0$. Hence, our original assertion that bids are strictly increasing in the players' own valuations is verified. The singularity at the point $v = 0$ doesn't really matter here since a player with realized valuation zero is actually just indifferent between participating or not in the auction.

[5] See problem JR #9.7 for a description of a first-price, all-pay auction.

$$\max_{b_i} \ (v_i - b_i)\Pr\left[b_i > b_j(v_j)\right] + (-b_i)\Pr\left[b_i < b_j(v_j)\right] \Leftrightarrow$$

$$\max_{b_i} \ -b_i + v_i \Pr\left[b_i > b_j(v_j)\right]$$

To solve for a BNE, suppose that player *j* adopts the strategy $b(.)$ and assume that $b(.)$ is strictly increasing and differentiable. Then for a given realization of v_i, player *i*'s optimal bid solves:

$$\max_{b_i} \ -b_i + v_i \Pr\left[b_i > b(v_j)\right] \qquad \text{(I)}$$

Let $b^{-1}(b_j) = b^{-1}(b(v_j)) = v_j$ the valuation that player *j* must have in order to be bidding b_j. Since $v_j \sim U[0,1]$ we have:

$$\begin{aligned} &-b_i + v_i \Pr\left[b_i > b(v_j)\right] \\ &= -b_i + v_i \Pr\left[b^{-1}(b_i) > b^{-1}(b(v_j))\right] \\ &= -b_i + v_i \frac{b^{-1}(b_i)}{(1-0)} \end{aligned}$$

Thus, the first order condition for player *i*'s optimization problem is:

$$-1 + v_i \frac{db^{-1}(b_i)}{db_i} = 0 \qquad \text{(III)}$$

The first order condition (III) is an implicit equation for bidder *i*'s best response to the strategy $b(.)$ played by bidder *j*, given that bidder *i*'s valuation has been realized as v_i. If we are looking for a symmetric BNE, we require that both bidders play the same strategy in equilibrium. Since, therefore, bidder *j* plays the strategy $b(.)$, this must be also played by bidder *i*, in equilibrium. Hence, we require that $b(.)$ is player *i*'s best response to $b(.)$ by player *j*. In other words, $b(v_i)$ must satisfy the first order condition (II): that is, for each of bidder *i*'s positive valuations, she does not wish to deviate from bidding according to the schedule $b(.)$, given that player *j* bids according to the same schedule.

To impose this requirement, we substitute $b_i = b(v_i)$ into (III):

$$v_i \frac{db^{-1}(b(v_i))}{db_i} = 1 \Leftrightarrow v_i \frac{dv_i}{db_i} = 1 \Leftrightarrow v_i \left(\frac{db_i}{dv_i}\right)^{-1} = 1$$

Our last equation must be viewed as a first-order differential equation that the function $b(.)$ must satisfy. Clearly, however, if this is to be satisfy for *any* values of v_i, v_j, it should be so for $v_i = v_j = v$. We now have:

$$\frac{v}{b'(v)} = 1 \Leftrightarrow b'(v) = v \Leftrightarrow b(v) = \frac{v^2}{2} + c$$

where c is the constant of integration.

To eliminate c we need a boundary condition. Fortunately, simple economic reasoning provides us with one: no player should bid more than his/her valuation. Thus, we require $b(v_i) \le v_i \quad \forall v_i \in [0,1]$. In particular, we require $b(0) \le 0$. Since bids are constrained to be non-negative, this implies that $b(0) = 0$.

Hence, $c = 0$ and our proposed BNE solution is that each bidder submits a bid according to the schedule: $b(v_i) = \frac{v_i^2}{2}$

To compare the optimal bids in the two auction settings, consider:

$b_i^{FPAP}(v_i) \ge b_i^{SPAP}(v_i) \Leftrightarrow \frac{v_i^2}{2} \ge -v_i - \ln(1-v_i) \Leftrightarrow \frac{v_i^2}{2} + v_i \ge -\ln(1-v_i)$ which clearly holds for $v_i \in [0,1]$ (see Fig. I).

Hence, in the second-price, al-pay auction each bidder bids lower than in the first-price, all-pay auction.

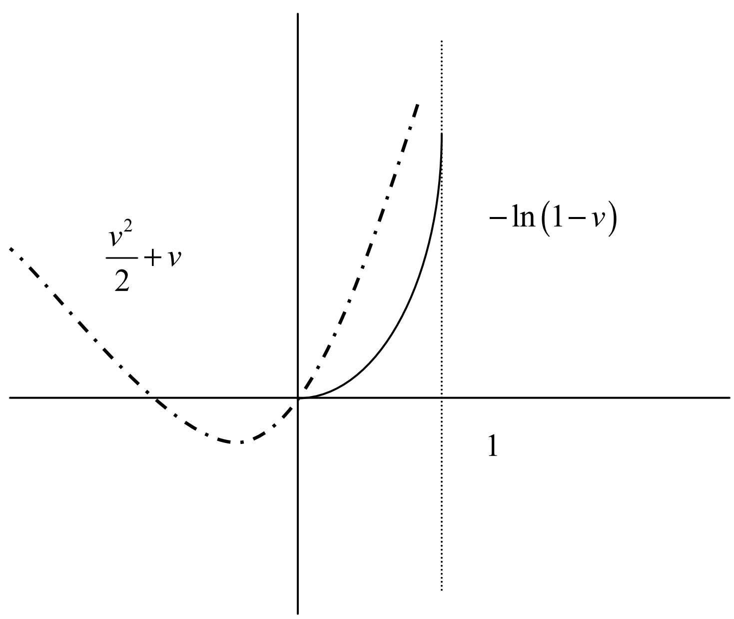

Fig. I

(c) The seller's expected revenue will be:

$$R^{SPAP} = 2E\left[b_j \middle| b_j < b_i\right] + 2E\left[b_i \middle| b_i < b_j\right]$$

$$= 2E\left[b(v_j) \middle| b(v_j) < b(v_i)\right] + 2E\left[b(v_i) \middle| b(v_i) < b(v_j)\right]$$

$$= 2E\left[b(v_j) \middle| v_j < v_i\right] + 2E\left[b(v_i) \middle| v_i < v_j\right]$$

$$= 2\int_{v_i=0}^{1}\left(\int_{v_j=0}^{v_i}\left(-v_j - \ln(1-v_j)\right)dv_j\right)dv_i + 2\int_{v_j=0}^{1}\left(\int_{v_i=0}^{v_j}\left(-v_i - \ln(1-v_i)\right)dv_i\right)dv_j$$

$$= 4\int_{v_i=0}^{1}\left(\int_{v_j=0}^{v_i}\left(-v_j - \ln(1-v_j)\right)dv_j\right)dv_i$$

$$= 4\int_{v_i=0}^{1}\left(\int_{v_j=0}^{v_i}\left(-v_j + \frac{d}{dv_j}\left((1-v_j)\ln(1-v_j)\right)+1\right)dv_j\right)dv_i$$

$$= 4\int_{v_i=0}^{1}\left(-\frac{v_j^2}{2} + (1-v_j)\ln(1-v_j) + v_j \Bigg|_{v_j=0}^{v_i}\right)dv_i$$

$$= 4\int_{v_i=0}^{1}\left(-\frac{v_i^2}{2} + (1-v_i)\ln(1-v_i) + v_i\right)dv_i$$

$$= 4\int_{v_i=0}^{1}\left(-\frac{v_i^2}{2} + v_i - \frac{d}{dv_j}\left(\frac{(1-v_i)^2}{2}\right)\ln(1-v_i)\right)dv_i$$

$$= 4\left\{-\frac{v_i^3}{6} + \frac{v_i^2}{2}\Bigg|_0^1 - \frac{(1-v_i)^2}{2}\ln(1-v_i)\Bigg|_0^1 - \int_{v_i=0}^{1}\frac{1-v_i}{2}dv_i\right\}$$

$$= 4\left\{-\frac{1}{6} + \frac{1}{2} - 0 + \frac{(1-v_i)^2}{4}\Bigg|_0^1\right\} = 4\left\{-\frac{1}{6} + \frac{1}{2} - \frac{1}{4}\right\} = \frac{1}{3}$$

(d) In a first-price auction, we know that each player bids optimally according to the bidding schedule (strategy): $b(v_i) = \frac{v_i}{2}$

The seller's expected revenue is given:

$$R^{FP} = E\left[b_i \middle| b_i > b_j\right] + E\left[b_j \middle| b_j > b_i\right] = 2E\left[b(v_i)\middle| b(v_i) > b(v_j)\right]$$

$$= 2E\left[\frac{v_i}{2}\middle| v_i > v_j\right]$$

$$= 2\int_{v_i=0}^{1}\left(\int_{v_j=0}^{v_i}\frac{v_i}{2}dv_j\right)dv_i = 2\int_{v_i=0}^{1}\frac{v_i}{2}\left[v_j\right]_0^{v_i}dv_i = 2\int_{v_i=0}^{1}\frac{v_i^2}{2}dv_i = \frac{v_i^3}{3}\bigg|_0^1 = \frac{1}{3}$$

Using the Revenue-Equivalence theorem, we note that:

1. Both auctions can be viewed as **incentive-compatible, direct-selling mechanisms**. The explanation as to why this is so for the first-price auction is given, in detail, in JR (pp. 383-384). Following exactly the same argument for the second-price, all-pay auction, it is easy to see that truthfully-revealing one's valuation is optimal. Consider the expected payoff to player I, in a direct-selling mechanism, when he reports that her type is r_i while the other player truthfully reports his own type to be v_j and bids are registered according to the second-price, all-pay BNE bidding function $b_i(x) = -x - \ln(1-x)$.
   $$-b(r_i) + \left(v_i + b(r_i) - b(v_j)\right)\Pr\left[b(r_i) > b(v_j)\right]$$
   We already know, from our analysis in part (a), that this payoff is maximized for $r_i = v_i$.
2. In both auctions, the probability assignment function is the same since the object is assigned, in equilibrium, to the player with the highest valuation. This is due to the fact that, in both auctions, the players' equilibrium bidding strategies are strictly increasing in the players' own valuations.
3. In both auctions, a bidder with zero valuation receives an equilibrium expected payoff of zero. Therefore, he is clearly indifferent between the two auction mechanisms.

(1)-(3) suffice for the theorem to apply. Consequently, the expected revenue to the seller ought to be the same between the first-price and the second-price all-pay auctions.

3. (JR #9.10)

This is a direct application of Theorem 9.9 in JR (pp. 399).

Given that each bidder *i*'s valuation v_i is drawn from the uniform distribution $U[0,1]$, assumption (9.22) on JR (pp. 394) is satisfied since:

$$v - \frac{1-F(v)}{f(v)} = v - \frac{1-v}{1} = 2v - 1$$

is clearly strictly increasing in v_i on the entire support of the distribution $U[0,1]$.

Therefore, the theorem asserts that a second-price, sealed bid auction with a reserve price of 0.5 is optimal. That is, the bidder with the highest bid strictly above 0.5 wins the object. Only the winner pays and she pays the maximum of the second highest bid and the reserve price. If no bids above 0.5 are placed, the seller keeps the object and no payments are made. Recall that, just as in the standard second-price, sealed bid auction without a reserve price, it is an optimal strategy, here, for each bidder to bid her own valuation.
For the reserve price we solved:

$$\rho - \frac{1-F(\rho)}{f(\rho)} = 0 \Leftrightarrow 2\rho - 1 = 0 \Leftrightarrow \rho = 0.5$$

4. (JR #9.11)
This again is a direct application of the Revenue Equivalence Theorem – Theorem 9.6 in JR (pp. 388).
Recall that, in employing the premises of the theorem, revenue equivalence depends only on: (1) the probability assignment functions and (2) the amount bidders have to pay when their values are zero.

1. In a first-price, sealed-bid auction with a reserve price of 0.5, the object goes to the bidder with the highest bid strictly above 0.5. Only the winner pays and he pays his bid. If no bids above 0.5 are placed, the seller keeps the object and no payments are made.
   It is easy to see that this auction has exactly the same probability assignment function as our optimal auction in the previous question. The object is assigned, in both auctions, to the bidder with the highest valuation and, since all bidders' values are drawn from the same distribution in both auctions, the probability of winning (i.e. being the bidder with the highest value) is the same. Moreover, in both auctions, if no bids are placed above 0.5, the object is kept by the seller. This occurs, in both auctions, if no bidder's value is drawn to be above 0.5. Again, since all bidders' values are drawn from the same distribution in both mechanisms, this occurs with the same probability in both auctions.
2. A bidder with zero value receives an expected payoff of zero in both auctions. Hence, he is indeed indifferent between the two mechanisms.

6. Let us label again the bidders by $i, j \in \{1,2\}$. Bidder *i* has valuation v_i for the good and the two bidders' valuations are independently distributed according to the following CDFs: $G(v_1) = v_1^{\alpha}$ and $G(v_2) = v_2^{\beta}$.
As usual, we assume that bids are constrained to be non-negative. Player *i*'s action is to submit a (non-negative) bid b_i, her type is her valuation v_i, her action space is $A_i = [0,\infty)$ and her type space is $T_i = (0,1)$. Note that, because the valuations are

independent, player *i* believes that v_j is distributed according to the CFD

$$G\left(v_j\right)=v_j^k : k=\begin{cases}\alpha & j=1\\ \beta & j=2\end{cases},$$

no matter what the realization v_i of her own valuation is.
Player *i*'s payoff function is:

$$u_i\left(b_1,b_2;v_1,v_2\right)=\begin{cases}v_i-b_i & i \text{ submits winning bid}\\ 0 & i \text{ submits losing bid}\end{cases}$$

The pair of strategies $\left(b_i\left(v_i\right),b_j\left(v_j\right)\right)$ constitutes a BNE here, if for each $v_i\in\left(0,1\right)$, $b_i\left(v_i\right)$ solves:

$$\max_{b_i}\ \left(v_i-b_i\right)\Pr\left[b_i>b_j\left(v_j\right)\right]$$

To solve for a BNE, suppose that player *j* adopts the strategy $b_j\left(.\right)$ and assume that $b_j\left(.\right)$ is strictly increasing and differentiable. Then for a given realization of v_i, player *i*'s optimal bid solves:

$$\max_{b_i}\ \left(v_i-b_i\right)\Pr\left[b_i>b\left(v_j\right)\right] \qquad \text{(I.1)}$$

Let $b_j^{-1}\left(b_j\right)=b_j^{-1}\left(b_j\left(v_j\right)\right)=v_j$, the valuation that player *j* must have in order to be bidding b_j. Since $G\left(v_j\right)=v_j^{\beta}$ we have:

$$\begin{aligned}\left(v_i-b_i\right)\Pr\left[b_i>b_j\left(v_j\right)\right]&=\left(v_i-b_i\right)\Pr\left[b_j^{-1}\left(b_i\right)>b_j^{-1}\left(b_j\left(v_j\right)\right)\right]\\ &=\left(v_i-b_i\right)\Pr\left[b_j^{-1}\left(b_i\right)>v_j\right]\\ &=\left(v_i-b_i\right)\left(b_j^{-1}\left(b_i\right)\right)^{\beta}\end{aligned}$$

Thus, the first order condition for player *i*'s optimization problem is:

$$-\left(b_j^{-1}\left(b_i\right)\right)^{\beta}+\beta\left(v_i-b_i\right)\left(b_j^{-1}\left(b_i\right)\right)^{\beta-1}\frac{db_j^{-1}\left(b_i\right)}{db_i}=0 \qquad \text{(II.1)}$$

The first order condition (II.1) is, as usual, an implicit equation for bidder *i*'s best response to the strategy $b_j\left(.\right)$ played by bidder *j* -given that bidder *i*'s valuation has been realized as v_i.
In this problem, we are looking to establish that each player bidding her conditional second value is a BNE. Consequently, we will take the bidding strategy of player *j* to be the expectation of player *i*'s valuation conditional upon *j*'s valuation being the highest of the two. Taking $i=1, j=2$ in (II.1), we have:

$$b_2(v_2)=\frac{E[v_1 \wedge v_1<v_2]}{\Pr[v_1<v_2]}=\frac{\int_{v_1=0}^{v_2} v_1 dG_1(v_1)}{G_1(v_2)}=\frac{\int_{v_1=0}^{v_2} v_1 d(v_1^{\alpha})}{v_2^{\alpha}}=\frac{\frac{\alpha}{\alpha+1}v_1^{\alpha+1}\Big|_0^{v_2}}{v_2^{\alpha}}=\frac{\alpha}{\alpha+1}v_2 \quad (1)$$

The first-order optimality condition (II.1) requires that b_1 is player 1's best response to $b_2(v_2)=\frac{\alpha}{\alpha+1}v_2$ by player 2. Noticing that $b_2^{-1}(z)=\frac{\alpha+1}{\alpha}z$, it is clear that, to impose this requirement, we need to substitute $b_2^{-1}(b_1)=\frac{\alpha+1}{\alpha}b_1$ into (II.1):

$$-\left(\frac{\alpha+1}{\alpha}b_1\right)^{\beta}+\beta(v_1-b_1)\left(\frac{\alpha+1}{\alpha}b_1\right)^{\beta-1}\left(\frac{\alpha+1}{\alpha}\right)=0 \Leftrightarrow$$

$$\beta(v_1-b_1)=b_1 \Leftrightarrow \quad b_1=\frac{\beta}{\beta+1}v_1$$

By looking at equation (1), it is obvious that our last equation dictates the best response of player 1, when player 2 is bidding according to his conditional second value, to be for player 1 to also bid according to her own conditional second value.

It should be now clear (although you should re-try it yourselves for practice) that, when player 1's strategy is to bid his conditional second value (i.e. $b_1(v_1)=\frac{\beta}{\beta+1}v_1$), player 2's best response is to adopt the same strategy (i.e. respond by bidding according to the schedule $b_2(v_2)=\frac{\alpha}{\alpha+1}v_2$) as this maximizes:

$$(v_2-b_2)\left(b_1^{-1}(b_2)\right)^{\alpha}=(v_2-b_2)\left(\frac{\beta+1}{\beta}b_2\right)^{\alpha} \qquad \text{(I.2)}$$

Hence, we have shown that, each player bidding her conditional second value, is indeed a BNE in this game.

7. First of all, notice that the way I am interpreting below the arrangement for the resolution of ties given in this problem is to mean the following:

(a) If we have a tie between the two bidders at a bid that is at least 300 (the seller's reservation price), the item goes to bidder 1

(b) If the highest bidder submits a bid of exactly 300, she takes the item.

Consider now the situation where each bidder adopts the strategy of bidding her conditional second value. Then the players will be submitting the following bids:

Player 1: $b_1(v_1) = 300$

Player 2: $b_2(v_2) = \begin{cases} 400 & v_2 = 800 \\ 300 & v_2 = 300 \end{cases}$

Clearly, when player 2 is bidding according to her conditional second value, the proposed strategy for player 1 is optimal.

- When playing against a type-v_2^H of player 2, player 1's bid does not win the object. Nevertheless, no other bid can actually improve his payoff in this event since winning requires bidding at least 400 - which is exactly the valuation of player 1 and bidding above one's results in a negative payoff in the case of a win. Bidding below 300 leaves the zero payoff from losing unchanged.
- When playing against a type-v_2^L of player 2, however, bidding 300 is strictly optimal for player 1. Due to the tie-breaking rules, player 1 wins the object and gets a payoff of 100. Any bid above 300 would still win the object but reduce the payoff whereas any bid below 300 loses the object forgoing the positive payoff of 100.

Hence, bidding her conditional second value is a best response for player 1 here by being a strict best response against the type-v_2^L of player 2 and weakly optimal (i.e. it is a best response but it is not uniquely so) against a type-v_2^H.

However, things do not look as good for player 2.

- Type-v_2^L of player 2 cannot do better than 300 against player 1 bidding 300. Bidding anything below 300 leaves the expected payoff unchanged at zero whereas bidding above 300 means bidding above type-v_2^L's own valuation.
- But type-v_2^H is playing sub-optimally at 400. She is currently winning the object and getting a payoff of 400. However, she can do strictly better by bidding $300+\varepsilon$ for $\varepsilon > 0$ and infinitesimally small. She would still win the object but improve her payoff by $100-\varepsilon$.

Therefore, bidding her conditional second value is not a best response for player 2 (since it is not so for one of her types) when player 1 is also bidding her own second value.

Clearly, this pair of bidding strategies cannot be a BNE.