Journal of Public Economics 46 (1991) 227–246. North-Holland

Steady-state solutions of optimal tax mixes in an overlapping-generations model

No-Ho Park*

Department of Economics, Stockholm University, S-106 91 Stockholm, Sweden

Received July 1989, revised version received March 1991

This paper analyzes steady-state solutions of optimal tax mixes in an overlapping-generations model of heterogeneous individuals with a utilitarian social welfare function. A test with Cobb–Douglas utility functions shows that an uneven distribution of the innate abilities leads to high rates of consumption and wage-income taxes, and a high level of a lump-sum transfer. The more the labor force works, the higher the optimal tax on wage income. Significant differences in individual preferences lead to low rates of consumption and wage-income taxes and to a high rate of interest-income tax. With identical preferences, the rate of interest-income tax is zero.

1. Introduction

The theory of optimal taxation is one of the oldest topics of public finance, going back at least to Dupuit (1844). In general, it can be separated into two categories: the theory of optimal consumption taxation and the theory of optimal income taxation. Originally, the studies centered on the theory of optimal consumption taxation. The first theoretical results appear in Ramsey (1927), which was the starting point of a great deal of studies on this subject and is well known today as the *Ramsey tax rule*. Subsequently, many papers on optimal consumption taxation have been published, with a peak in the 1970s. At this time, studies also started to appear on the theory of optimal income taxation.[1]

The purpose of this paper is to derive an optimal consumption and income tax mix for an economy with heterogeneous individuals and distributional objectives, allowing the government to run a deficit financed by issuing bonds. For that purpose, we use the Diamond (1965) version of Samuelson's

*I am grateful to my supervisor, Professor Peter Bohm, for his valuable and stimulating comments and criticisms on earlier versions of the paper. I also wish to thank Professors K.G. Jungenfelt and T. Persson, and two anonymous referees for their helpful comments.

[1]For a clear introduction to the theory of optimal taxation and the literature, see Sandmo (1976) or Stern (1984).

(1958) intergenerations model of neoclassical growth. Of course, the optimal tax mix may depend significantly on the distributional objectives of the government and on the range of taxes at the disposal of the government. Here, we allow a wide range of taxes to be used: a consumption tax, a wage- and an interest-income tax, a wealth tax, and a payroll tax. These taxes account for a significant share of the actual government tax revenue in modern industrialized countries. For example, in 1985 the share amounted to 89.7 percent in Sweden, 82.8 percent in the United States, 75.1 percent in the United Kingdom, and 69.0 percent in Japan.[2] In addition, we allow the government to use a lump-sum transfer.

There have been very few studies of the optimal tax mix in an overlapping-generations setting. Ordover and Phelps (1975) discussed the optimal mix of linear taxes of wealth and wages that maximize a maximin social welfare function. Ordover (1976) studied the optimal mix of linear taxes on wages and interest. Ordover and Phelps (1979) discussed taxes on capital and wealth permitting each generation to employ graduated (non-linear) taxes on interest and wage earnings. Recently, Svensson and Weibull (1987) analyzed optimal linear taxes on labor and capital income in combination with a lump-sum transfer. However, in all of these papers, consumption taxes are assumed to be zero. In Atkinson and Stiglitz (1980), a model for an optimal tax mix is formulated assuming identical individuals and incorporating such taxes as a consumption tax, a wage- and an interest-income tax, and a lump-sum tax. Even though they started out with a non-zero consumption tax, they derived results only for wage- and interest-income taxes, saying that they 'can without loss of generality set the consumption tax at zero' (p. 445). In the present paper, an optimal tax mix is derived for heterogeneous individuals, allowing a non-zero consumption tax.

In section 2 we formulate a fairly general model of a general-equilibrium, overlapping-generations economy with two-period-lived heterogeneous individuals, with the gross wage rates and the gross interest rate being endogenous and using the principle of optimality of dynamic programming. Since the analysis of the general model provides no specific results, we introduce in section 3 Cobb–Douglas utility functions to obtain a more intuitive notion, referring further analysis of the general model to the appendix. The distributional objectives of the government are characterized by a utilitarian social welfare function, which is an unweighted sum of individual utilities. Finally, section 4 summarizes the main points.

2. The general model

The closed economy considered here has an infinite future of discrete time

[2]U.S. Bureau of the Census (1987, p. 810).

and consists of overlapping generations of two-period-lived heterogeneous individuals, one firm and a government. By 'heterogeneous individuals' we mean that individuals differ in their preferences and innate abilities to produce. The population of generation s is denoted by N_s, and its rate of growth over generations is exogenously given as n. Individuals born in period s are said to be of generation s and are retired from work in period $s+1$.

Individual i of generation s supplies labor ℓ_s^i at a wage rate of w_s^i per unit of time in the first period and consumes c_s^i and z_{s+1}^i in the first and the second periods. Provided that individuals' wage rates indicate their innate production abilities, we can define l_s^i as the *effective* labor supplied by individual i of generation s in a manner such that $w_s l_s^i \equiv w_s^i \ell_s^i$, with w_s being the average wage rate of generation s. This normalization implies that the effective-labor supplies are weighted actual-labor supplies, with the weights being the ratio of individual wage rates to the average wage rate. The endowment of individual i's effective time is exogenously given and is denoted by e_s^i. The more skilful an individual is at work, the higher the level of e^i would be. Individual i of generation s saves for his second-period consumption, and the amount of his savings at the end of the first period is denoted by a_s^i. No one bequeaths anything to coming generations.

A wage-income tax with a proportional rate of t_s^w is imposed and a lump-sum transfer is provided to every individual of generation s by an equal amount of m_s in period s.[3] We assume that there is only one kind of consumption good and that the producer price of the good is unity, making the consumption good the numeraire. In principle, different rates of indirect taxes could be imposed on the first- and second-period consumption of the good so that individuals of generation s meet with consumption taxes at a rate of t_s in the first period and τ_{s+1} in the second period. The rate of return to capital, accumulated in period s and held over to period $s+1$, is denoted by r_{s+1} and taxed at a rate of t_{s+1}^r in period $s+1$. The asset that individual i of generation s possesses in the beginning of the period $s+1$ equals his savings multiplied by a factor of 1 plus the net interest rate. A wealth tax of rate t_{s+1}^a is imposed on this asset.

Now, the first- and second-period budget equations of individual i of generation s are

$$a_s^i = (1-t_s^w) w_s l_s^i + m_s - (1+t_s) c_s^i, \tag{1a}$$

$$(1-t_{s+1}^a)[1+(1-t_{s+1}^r) r_{s+1}] a_s^i = (1+\tau_{s+1}) z_{s+1}^i, \tag{1b}$$

[3]It is also possible to include a further lump-sum transfer in period 2. However, it can be shown to be equivalent to the issue of government debt. See Atkinson and Sandmo (1980, p. 533).

which in turn gives his lifetime budget restriction such that

$$(1+t_s)c_s^i + p_{s+1}z_{s+1}^i - \omega_s l_s^i = m_s, \tag{2}$$

where $p_{s+1} \equiv (1+\tau_{s+1})q_{s+1}$, with $q_{s+1} \equiv [(1-t_{s+1}^{\mathrm{a}})\{1+(1-t_{s+1}^{\mathrm{r}})r_{s+1}\}]^{-1}$. We can interpret $(1+t_s)$ and p_{s+1} as the consumer price in periods 1 and 2, respectively, and $\omega_s \equiv (1-t_s^{\mathrm{w}})w_s$ as the after-tax average rate of wages.

We assume there is no uncertainty. Individual i of generation s maximizes his own utility function,

$$u_s^i = U_s^i(c_s^i, z_{s+1}^i, l_s^i), \tag{3}$$

by choosing consumption in the two periods and his supply of effective labor in the first period of his life, subject to his lifetime budget restriction, eq. (2). We can then write the consumption functions and the effective-labor supply function for individual i of generation s as depending on t_s, p_{s+1}, ω_s, and m_s. Reintroducing these functions in eq. (3), we can define an indirect utility function such that

$$v_s^i = V_s^i(t_s, p_{s+1}, \omega_s, m_s). \tag{4}$$

Production in period s is represented by a neoclassical, one-product, constant-returns-to-scale (CRS) production function, $F(K_s, L_s^{\mathrm{d}})$, relating total output to aggregate effective labor employed in period s, L_s^{d}, and the capital stock determined in the preceding period, K_s. The form of this production function is assumed to be identical for all generations. The CRS property makes $\mathrm{F}(k_s, l_s^{\mathrm{d}})$ the output per capita of generation s, where $k_s \equiv K_s/N_s$ and $l_s^{\mathrm{d}} \equiv L_s^{\mathrm{d}}/N_s$. We assume there is no government production or consumption. We also assume that there is neither technical progress nor depreciation. Then the capital stock in period $s+1$ is determined by the capital stock employed in period s *plus* the savings of generation s *less* the budget deficit of government s. However, since there are offsetting dissavings by the retired generation, the capital stock in period $s+1$ equals the savings of generation s *less* the budget deficit of government s, provided that there was neither any capital stock nor any government deficit at the outset of the economy. Hence,

$$K_{s+1} \equiv N_s(a_s - b_s),$$

where b_s denotes the per capita government deficit at the end of period s.[4] We use such terms as c, z, e, l, a, b, k, w and ω without any individual index

[4]Since there was no capital stock or government deficit in period 0, the capital stock in period 1 was $K_1 = N_0 a_0$. In period 2, it is $K_2 \equiv K_1 + N_1(a_1 - b_1) - N_0 a_0 = N_1(a_1 - b_1)$, and in period $s+1$, $K_{s+1} \equiv K_s + N_s(a_s - b_s) - N_{s-1}(a_{s-1} - b_{s-1}) = N_s(a_s - b_s)$.

for average or per capita terms. Using eq. (1a), we can rewrite this condition in per capita form such that

$$(1+n)k_{s+1}=\omega_s l_s+m_s-(1+t_s)c_s-b_s. \tag{5}$$

The firm, behaving like a pricetaker, pays a payroll tax at a rate of t_s^{f} per wage payments in period s. The profit-maximizing problem of the firm is

$$\max \Pi_s=N_s[F(k_s,l_s^{\mathrm{d}})-(1+t_s^{\mathrm{f}})w_s l_s^{\mathrm{d}}-(1+r_s)k_s],$$

and this maximizing process yields the factor price frontier:

$$1+r_s=F_k(k_s,l_s^{\mathrm{d}}), \tag{6a}$$

$$(1+t_s^{\mathrm{f}})w_s=F_l(k_s,l_s^{\mathrm{d}}), \tag{6b}$$

where $F_k(\cdot)$ and $F_l(\cdot)$ denote the marginal products of capital and effective labor, respectively. Since there is no government consumption, the production in period s is consumed by the present generation and the retired generation who are a factor $(1+n)^{-1}$ fewer, with the rest being invested in the next period. Thus in per capita form we have

$$F(k_s,l_s^{\mathrm{d}})=c_s+\frac{z_s}{1+n}+(1+n)k_{s+1}. \tag{7}$$

The government in period s, called government s, collects revenues from generations s and $s-1$, and from the firm in period s. On the other hand, the total expenditure of government s consists of the lump-sum transfer to generation s and the principal and interest payments on the bonds issued by the government $s-1$. Government s finances its deficit by issuing bonds. Hence, we can write the government budget restriction in per capita form as

$$b_s=m_s+\frac{1+r_s}{1+n}b_{s-1}-[t_s c_s+t_s^{\mathrm{w}}w_s l_s+t_s^{\mathrm{f}}w_s l_s^{\mathrm{d}}]$$

$$-\frac{1}{1+n}[\tau_s z_s+t_s^{\mathrm{r}}r_s a_{s-1}+t_s^{\mathrm{a}}\{1+(1-t_s^{\mathrm{r}})r_s\}a_{s-1}]. \tag{8}$$

Depending on taxes and the lump-sum transfer, equilibrium gross wage rates and the gross interest rate are determined by the following equilibrium conditions:

$$(1+n)k_{s+1}=\omega_s l_s+m_s-(1+t_s)c_s-b_s, \tag{5}$$

$$F(k_s, l_s^{\mathrm{d}})=c_s+\frac{1}{1+n}z_s+(1+n)k_{s+1}, \tag{7}$$

$$l_s^{\mathrm{d}}=l_s. \tag{9}$$

Eqs. (5), (7) and (9) represent equilibria in the capital market, the product market, and the effective-labor market, respectively. These conditions, together with individual budget restrictions, imply the government budget restriction, eq. (8), by Walras' law. Substituting eq. (9) into (7) and eliminating k_{s+1} from eqs. (5) and (7) gives

$$F(k_s, l_s)+t_s c_s+b_s-\frac{1}{1+n}z_s-\omega_s l_s-m_s=0. \tag{10}$$

This equation is a compact expression of the equilibrium conditions for the three markets specified by eqs. (5), (7) and (9).

As defined here, the wealth tax and the interest-income tax are interchangeable, and the same is true for the payroll tax and wage-income tax. Thus, we can set the rates of wealth tax and payroll tax at zero without any loss of generality. Now, the policy parameters at the disposal of government s are the consumption tax rate in period s and $s+1$, t_s and τ_{s+1}; the rate of interest-income tax, t_{s+1}^{r}; the rate of wage-income tax, t_s^{w}; and the level of lump-sum transfer in period s, m_s.

Government s aims at maximizing the level of social welfare only of generation s, with the parameters that determine the welfare of the preceding generation being fixed. This implies that government s is committed to fulfill the policies determined by the previous government. The level of social welfare of generation s is represented by a function $\Omega_s(V(s))$ of $(1\times N_s)$ row vector of indirect utilities of individuals of generation s, $V(s)$, with Ω_s/N_s denoting the average level. We introduce a state valuation function at state s, W_s, which is a weighted sum of maximized present and discounted future level of average welfare of individuals in each generation in a manner such that

$$W_s(k_s, t_{s-1}, \tau_s, t_s^{\mathrm{r}}, t_{s-1}^{\mathrm{w}}, m_{s-1})=\max\sum_{j=s}^{\infty}\left(\frac{1}{1+\rho}\right)^{j-s}\left(\frac{\Omega_j}{N_j}\right),$$

subject to eq. (10). ρ is the social rate of time preference and is assumed to be a positive constant. Since the conditions of the so-called convergence results are satisfied, we can write the Bellman equation using the principle of optimality of dynamic programming[5] as follows:

$$
\begin{aligned}
&W(k_s, t_{s-1}, \tau_s, t_s^{\mathrm{r}}, t_{s-1}^{\mathrm{w}}, m_{s-1}) \\
&\quad = \max\left[\left\{\frac{\Omega_s(V(s))}{N_s}\right\}\right. \\
&\qquad + \mu_s\left\{F(k_s, l_s) + t_s c_s + b_s - \frac{z_s}{1+n} - \omega_s l_s - m_s\right\} \\
&\qquad \left. + \frac{1}{1+\rho} W(k_{s+1}, t_s, \tau_{s+1}, t_{s+1}^{\mathrm{r}}, t_s^{\mathrm{w}}, m_s)\right],
\end{aligned}
\tag{11}
$$

where k_{s+1} is given by eq. (7). Suppose that b_s is freely variable. Since b_s does not enter the individuals' indirect utility functions, the necessary condition for optimality is $\mu_s = 0$. The debt policy allows the government to influence the relation between private savings and the level of capital formation. In other words, this debt policy ensures that a level of k_{s+1} that is feasible according to the production constraint, eq. (7), can be achieved by individual decisions subject to their lifetime budget constraint, eq. (5).

3. Steady-state solutions with Cobb–Douglas utility functions

Referring further analysis of the general model just presented to the appendix at the end of the paper, in order to derive specific results we concentrate here on a case of Cobb–Douglas (C–D for short) utility function of a type

$$u^i = \sigma_1^i \ln c^i + \sigma_2^i \ln z^i + \sigma_3^i \ln(e^i - l^i), \tag{12}$$

where $\sigma_1^i + \sigma_2^i + \sigma_3^i = 1$ for all individuals.

The central properties of C–D utility functions are

$$c^i = \frac{1}{1+t}\sigma_1^i y^i, \tag{13a}$$

[5]See, for example, Bellman and Dreyfus (1962) or Intriligator (1971).

$$z^i = \frac{1}{p}\sigma_2^i y^i, \tag{13b}$$

$$l^i = e^i - \frac{1}{\omega}\sigma_3^i y^i = (1-\sigma_3^i)e^i - \frac{1}{\omega}\sigma_3^i m, \tag{13c}$$

where $y^i \equiv (\omega e^i + m)$ is the sum of the after-tax market value of the total endowment of effective time of individual i and the lump-sum transfer, which we call his lifetime endowment of disposable income (LEDI for short). A high innate ability ensures a high level of LEDI. The terms σ_1^i, σ_2^i and σ_3^i represent individual i's preferences in the sense that he expends specific shares of his LEDI on first- and second-period consumption, and on leisure measured in effective terms, respectively. The Marshallian versions of consumption functions and the effective-leisure demand function are functions of own price and LEDI. Moreover, the property of the C–D utility function says that the inverse of LEDI represents the marginal utility of income. Thus, the more able individuals show lower levels of marginal utility of income than the less able individuals.

By the same process as in the general model presented in the appendix, we obtain the *modified golden rule*,

$$1+r = (1+n)(1+\rho),$$

and the following equation system equivalent to eqs. (A.4) of the general model:

$$\sum_i \left[\alpha^i - \frac{\phi}{1+t}\right]\frac{1}{1+t}\sigma_1^i y^i = 0, \tag{14a}$$

$$\sum_i \left[\alpha^i - \frac{\phi}{p(1+r)}\right]\frac{1}{p}\sigma_2^i y^i = 0, \tag{14b}$$

$$\sum_i \left[\alpha^i - \frac{\phi}{1-t^w}\right]\frac{1}{\omega}\sigma_3^i y^i$$

$$= \sum_i \left[\alpha^i - \phi\left(\frac{1}{1+t}\sigma_1^i + \frac{1}{p(1+r)}\sigma_2^i + \frac{1}{1-t^w}\sigma_3^i\right)\right]e^i, \tag{14c}$$

$$\sum_i \left[\alpha^i - \phi \left(\frac{1}{1+t} \sigma_1^i + \frac{1}{p(1+r)} \sigma_2^i + \frac{1}{1-t^{\mathrm{w}}} \sigma_3^i \right) \right] = 0, \tag{14d}$$

where the term α^i, defined as $\alpha^i \equiv \beta^i \lambda^i$ and referred as to the marginal social utility of income accruing to individual i, is a product of the marginal contribution of individual i's utility to the social welfare, $\beta^i \equiv (\partial\Omega/\partial v^i)$, and the marginal private utility of income of individual i, $\lambda^i \equiv (\partial v^i/\partial m)$. The term ϕ is defined as $\phi \equiv [W_1/(1+n)(1+\rho)]$ with $W_1 \equiv \partial W/\partial k$, and interpreted as the marginal social valuation of capital accumulation.

The marginal contribution of individual i's utility to the social welfare, β^i, characterizes the distributional objectives of the government. It is the government that evaluates the level of β^i. The government is now confronted with a value judgment of the term β^i for deciding its policy instrument. In this context, the distributional objective is a prerequisite for determining the levels of optimal tax rates. If the government sets α^i to be constant and equal to unity for all individuals, this would imply that it has no distributional objectives. In this case, we have $t = \tau = t^{\mathrm{r}} = t^{\mathrm{w}} = 0$ and $m = -\rho b$.

Here, we analyze a case where the government has distributional objectives, with $\beta^i = 1$ for all individuals. Then, we have $\alpha^i = \lambda^i$. This assumption implies a utilitarian social welfare function, i.e. the social welfare function is an unweighted sum of individual utilities. In other words, the government is indifferent between marginal increases in the well-being of individuals with low innate abilities and that of individuals with high innate abilities. Hence, a one-dollar increase in income to the less able individuals contributes more to the social welfare than the same increase in income to the more able individuals, since the less able individuals have a higher level of marginal utilities of income than the more able individuals. Thus, the government can achieve a larger increase in the social welfare from a one-dollar lump-sum transfer to the less than to the more able individuals, and less of a decrease in the social welfare from a one-dollar lump-sum tax on the more than on the less able individuals. Assuming that this characterizes the distributional objectives of the government, we obtain a reason why the government may want to use non-lump-sum taxes.

Recall that we have defined the marginal social valuation of capital accumulation as $\phi \equiv [W_1/(1+n)(1+\rho)]$, with $W_1 \equiv \partial W/\partial k$. In the steady state, it is a reasonable assumption that the term ϕ equals the average marginal social utility of income, i.e. $\phi = \alpha$. This assumption, together with the assumption of $\beta^i = 1$ and the property of the C–D utility function, $\lambda^i = 1/y^i$, implies that

$$\alpha = \phi = \lambda = E[1/y], \tag{15}$$

where $E[1/y]$ denotes the average level of inverses of LEDI, while $E[y]$ denotes the average level of LEDI. Substituting eq. (15) into eq. (14d), we obtain the following result:

$$\frac{1}{1+t}\sigma_1+\frac{1}{p(1+r)}\sigma_2+\frac{1}{1-t^{\mathrm{w}}}\sigma_3=1. \tag{16}$$

Using eq. (15), we can rewrite eqs. (14) as follows:

$$1+t=\theta(\mathrm{cov}_1+1), \tag{17a}$$

$$p(1+r)=\theta(\mathrm{cov}_2+1), \tag{17b}$$

$$1-t^{\mathrm{w}}=\sigma_3\left[1-\frac{\sigma_1}{\theta(\mathrm{cov}_1+1)}-\frac{\sigma_2}{\theta(\mathrm{cov}_2+1)}\right]^{-1}, \tag{17c}$$

where $\mathrm{cov}_1 \equiv \mathrm{cov}(\sigma_1^i/\sigma_1, y^i/E[y])$ is a normalized covariance between individual preference on the first-period consumption and the LEDI, and similarly for cov_2 and cov_3. The levels of these covariances are confined to $(-1\ 1)$. Since $\sigma_1^i+\sigma_2^i+\sigma_3^i=1$ for all individuals, it is clear that $\sigma_1\,\mathrm{cov}_1+\sigma_2\,\mathrm{cov}_2+\sigma_3\,\mathrm{cov}_3=0$. If individuals were identical, we would have covariances equal to zero.[6] Thus, the heterogeneity of individuals is a necessary condition for non-zero covariances. The term $\theta \equiv E[y]E[1/y] \approx E[e]E[1/e]$ characterizes the distribution of LEDI among individuals, which in turn reflects the distribution of the innate abilities of the labor force. It is greater than unity, unless $e^i=e$ for all individuals. This implies that it is larger in an economy where the innate abilities of the members of the labor force vary significantly than in an economy where all members of the labor force are almost equally skilful.

If we still had a non-zero wealth tax, eq. (17b) would be a solution with $p \equiv (1-\tau)[(1-t^{\mathrm{a}})\{1+(1-t^{\mathrm{r}})r\}]^{-1}$, and with a non-zero payroll tax eq. (17c) would be a solution to $(t^{\mathrm{w}}+t^{\mathrm{f}})/(1-t^{\mathrm{w}})$. Since the term $p(1+r)$ in eq. (17b) is a combination of the rate of second-period consumption tax, τ, and the rate of interest-income tax, t^{r}, we cannot solve explicitly for these two tax rates. This in turn implies that the second-period consumption tax is interchangeable with the interest-income tax. Since it is technically difficult, though not impossible, to levy two different rates of consumption tax in the same period, $\tau \neq t$, a reasonable requirement would be that consumption should be taxed at only one rate, i.e.

[6]For the case of $t=\tau=m=0$ with identical individuals, see Atkinson and Stiglitz (1980, pp. 442–451).

$\tau \equiv t.$

Substituting eqs. (17) into the government budget restriction, and using the *modified golden rule*, we obtain the optimal level of lump-sum transfer, *m*. Then, the explicit solution is

$$t = \theta(\mathrm{cov}_1 + 1) - 1, \tag{18a}$$

$$t^{r} = \frac{(1+n)(1+\rho)}{\{(1+n)(1+\rho)-1\}} \frac{(\mathrm{cov}_2 - \mathrm{cov}_1)}{(\mathrm{cov}_2 + 1)}, \tag{18b}$$

$$t^{w} = 1 - \sigma_3 \left[1 - \frac{\sigma_1}{\theta(\mathrm{cov}_1 + 1)} - \frac{\sigma_2}{\theta(\mathrm{cov}_2 + 1)} \right]^{-1}, \tag{18c}$$

$$\begin{aligned} m &= \left[\frac{1}{\Delta} - (1 - t^{w}) \right] we - \frac{1}{\Delta} \rho b \\ &= \left[\frac{1}{\Delta} - \sigma_3 \left\{ 1 - \frac{\sigma_1}{\theta(\mathrm{cov}_1 + 1)} - \frac{\sigma_2}{\theta(\mathrm{cov}_2 + 1)} \right\}^{-1} \right] we - \frac{1}{\Delta} \rho b, \end{aligned} \tag{18d}$$

where

$$\begin{aligned} \Delta \equiv & \left[1 - \frac{\sigma_1}{\theta(\mathrm{cov}_1 + 1)} - \frac{\sigma_2}{\theta(\mathrm{cov}_2 + 1)} \right] (\mathrm{cov}_3 + 1) \\ & + \frac{1}{\theta}(1 - \sigma_3) - \rho \sigma_2 \left[(\mathrm{cov}_2 + 1) - \frac{1}{\theta} \right]. \end{aligned}$$

The solutions in eqs. (18) say that, with C–D utility functions, the optimal tax mix depends on the distribution of innate abilities and the structure of individual preferences. Although it is an empirical matter to estimate the correlations between preferences and LEDI, we assume here that the more able individuals show lower preferences for first-period consumption and higher preferences for second-period consumption and leisure than the less able individuals. That is, the first-period consumption is directed toward necessities and the second-period consumption and leisure toward luxury goods. Then, cov_1 has a negative value while cov_2 and cov_3 are positive.

If individuals are wholly identical, the rates of non-lump-sum taxes are all zero and the lump-sum transfer will be a lump-sum tax which, together with the new issue of debt, finances the principal and interest payments on the existing bonds. Hence, the rationale for employing non-lump-sum taxes lies in the heterogeneity of individuals. If individuals show identical preferences, the rate of interest-income tax is zero, and the rates of consumption tax and the wage-income tax depend on the distribution of the individual innate abilities.

With heterogeneous individuals, the rate of consumption tax is increasing with the term θ. This implies that the greater the variations in the innate abilities of the labor force, the higher the rate of consumption tax. Since cov_1 is assumed to be negative, in an economy where the preferences with respect to the first-period consumption vary significantly, the rate of consumption tax should be low. The rate of interest-income tax is positive. However, it does not depend on the variations in the individual innate abilities but on the relative sizes of covariances, the social rate of time preference, and the population growth rate. It is clear that the differences in the individual preferences have a dramatic effect on the rate of interest-income tax. Thus, the interest-income tax is not an efficient policy parameter to achieve the distributional objectives. Since the *modified golden rule* holds, the lower the population growth or time preference, the higher the rate of taxes on interest earnings. On the other hand, the rate of wage-income tax is increasing with the term θ and decreasing with the term σ_3. Thus, in an economy where the innate abilities of the labor force vary significantly and the more the labor force works, the higher the tax rate that should be imposed on wage income. The lump-sum transfer is also increasing with the term θ.

Suppose, for example, $\sigma_1=0.4$, $\sigma_2=\sigma_3=0.3$, $\mathrm{cov}_1=-0.1$, $\mathrm{cov}_2=0.1$, $\mathrm{cov}_3=1/30$ so that $\sigma_1\mathrm{cov}_1+\sigma_2\mathrm{cov}_2+\sigma_3\mathrm{cov}_3=0$, and $\theta=1.2$, $n=0.05$, $\rho=0.5$. Then, we have $t=0.08$, $t^r=0.498$, $t^w=0.254$, and $\Delta=0.959$. With identical preferences, they will be $t=0.2$, $t^r=0$, $t^w=0.280$, and $\Delta=0.975$. With wholly identical individuals, $t=t^r=t^w=0$, $\Delta=1$, and $m=-b/2$.

To sum up, in an economy where the innate abilities of the labor force vary significantly, a relatively large amount of revenues should be collected using a consumption tax and a wage-income tax. Then, together with the issue of new debt, the revenues from the non-lump-sum taxes finance, in addition to the repayment of existing debt, the high level of lump-sum transfers to the present generation, achieving the distributional objectives of the government. The rate of interest-income tax is positive. However, it does not depend on the variations in the individuals' innate abilities but on the relative sizes of covariances, the social rate of time preference, and the population growth rate, implying that the interest-income tax is not an efficient policy parameter to achieve the distributional objectives. Since the *modified golden rule* holds, the lower the population growth or time preference, the higher the rate of taxes on interest earnings. On the other hand, significant variations in the preferences with respect to the first-period consumption lead to low levels of consumption tax and wage-income tax and to a high level of interest-income tax. If the preferences with respect to the second-period consumption vary significantly, this leads to a low level of wage-income tax and a high level of interest-income tax. Moreover, the more the labor force works in an economy, the higher the tax rate that should be imposed on wage income.

4. Concluding remarks

We have studied the steady-state level of the optimal mix of direct and indirect taxes incorporating a consumption tax, taxes on wage and interest earnings, a wealth tax, and a payroll tax in combination with a lump-sum transfer in an overlapping-generations setting. We assumed a utilitarian social welfare function, which implies that the social welfare function is an unweighted sum of individuals' indirect utilities. Using the principle of optimality of dynamic programming, we explicitly derived the formula for the optimal mix of taxes and the lump-sum transfer. With the general model, the results of which are presented in the appendix, the rates of taxes are highly sensitive to the compensated elasticities and covariances, and it is especially problematical that we have little empirical data on some of these parameters. Owing to this lack of empirical information, it has not been possible to interpret the general results in a more exhaustive way. However, we have found that, ceteris paribus, the lower the population growth or the time preference, the higher the rate of taxes on interest earnings.

To derive specific results we assumed Cobb–Douglas utility functions and found that the optimal mix of taxes and the lump-sum transfer depends on the distribution of innate abilities and the structure of individual preferences. It is assumed there that the first-period consumption is of necessities and that the the second-period consumption and leisure are luxury goods. The rate of interest-income tax is positive depending on the relative sizes of covariances, the social rate of time preference and the population growth rate, implying that the interest-income tax is not an efficient policy parameter to achieve the distributional objectives. In an economy where the innate abilities of the labor force vary significantly, relatively large amounts of revenues should be collected by the consumption tax and the wage-income tax. Then, together with the issue of new debt, the revenues from the non-lump-sum taxes finance, in addition to the repayment of existing debt, the high level of lump-sum transfer to the present generation, achieving the distributional objectives of the government. On the other hand, significant variations in the preferences for first-period consumption lead to low levels of consumption tax and wage-income tax and to a high level of interest-income tax. If the preferences for second-period consumption vary significantly, the result would be a low level of wage-income tax and a high level of interest-income tax. Moreover, the more the labor force works in an economy, the higher the tax rate that should be imposed on wage income. The distortions in the economy due to these non-lump-sum taxes are the opportunity costs of the distributional objectives of the government. In other words, these distortions represent a kind of trade-off between efficiency and equity.

Appendix: Analysis of the general model

We can obtain the first-order conditions by differentiating the Bellman equation, eq. (11), with respect to t_s, τ_{s+1}, s^{r}_{s+1}, t^{w}_s, and m_s:

$$-\sum_i \beta^i \frac{\mathrm{d}V^i}{\mathrm{d}t} = \phi \sum_i [(wl^i_t - c^i_t) + (wl^i_p - c^i_p)p_t + (wl^i_\omega - c^i_\omega)\omega_t] + \frac{1}{1+\rho} N_s W_2(s+1), \tag{A.1a}$$

$$-\sum_i \beta^i \frac{\mathrm{d}V^i}{\mathrm{d}\tau} = \phi \sum_i [(wl^i_p - c^i_p)p_\tau + (wl^i_\omega - c^i_\omega)\omega_\tau] + \frac{1}{1+\rho} N_s W_3(s+1), \tag{A.1b}$$

$$-\sum_i \beta^i \frac{\mathrm{d}V^i}{\mathrm{d}t^{\mathrm{r}}} = \phi \sum_i [(wl^i_p - c^i_p)p_{t^{\mathrm{r}}} + (wl^i_\omega - c^i_\omega)\omega_{t^{\mathrm{r}}}] + \frac{1}{1+\rho} N_s W_4(s+1), \tag{A.1c}$$

$$-\sum_i \beta^i \frac{\mathrm{d}V^i}{\mathrm{d}t^{\mathrm{w}}} = \phi \sum_i [(wl^i_p - c^i_p)p_{t^{\mathrm{w}}} + (wl^i_\omega - c^i_\omega)\omega_{t^{\mathrm{w}}}] + \frac{1}{1+\rho} N_s W_5(s+1), \tag{A.1d}$$

$$-\sum_i \beta^i \frac{\mathrm{d}V^i}{\mathrm{d}m} = \phi \sum_i [(wl^i_p - c^i_p)p_m + (wl^i_\omega - c^i_\omega)\omega_m + (wl^i_m - c^i_m)] + \frac{1}{1+\rho} N_s W_6(s+1), \tag{A.1e}$$

where the summation runs from 1 to N_s, and the period indices are partially omitted. $W_j(s)$ denotes the partial derivative of the state valuation function at state s with respect to the jth argument. $\beta^i \equiv (\partial\Omega/\partial v^i)$ is the level of the marginal contribution of individual i's utility to the social welfare. The term $\phi \equiv [W_1(s+1)/(1+n)(1+\rho)]$ is the marginal social valuation of capital accumulation. For the derivation of eqs. (A.1) we have used the first-order conditions of the profit-maximizing firm, eqs. (6), and the effective-labor market equilibrium condition, eq. (9).

Moreover, we can obtain a series of difference equations governing the state valuation function by differentiating the Bellman equation with respect to t_{s-1}, τ_s, t^{r}_s, t^{w}_{s-1}, m_{s-1}, and k_s:

$$N_s W_2(s) = -\frac{\phi}{1+n}\sum_i [z^i_t + z^i_p p_t + z^i_\omega \omega_t], \tag{A.2a}$$

$$N_s W_3(s) = -\frac{\phi}{1+n}\sum_i [z^i_p p_\tau + z^i_\omega \omega_\tau], \tag{A.2b}$$

$$N_s W_4(s) = -\frac{\phi}{1+n}\sum_i [z^i_p p_{t^{\mathrm{r}}} + z^i_\omega \omega_{t^{\mathrm{r}}}], \tag{A.2c}$$

$$N_s W_5(s) = -\frac{\phi}{1+n}\sum_i [z^i_p p_{t^{\mathrm{w}}} + z^i_\omega \omega_{t^{\mathrm{w}}}], \tag{A.2d}$$

$$N_s W_6(s) = -\frac{\phi}{1+n}\sum_i [z^i_p p_m + z^i_\omega \omega_m + z^i_m], \tag{A.2e}$$

$$N_s[W_1(s) - \phi(1+r)] - \sum_i \beta^i \frac{\mathrm{d}V^i}{\mathrm{d}k}$$

$$= \phi \sum_i \left[(wl^i_p - c^i_p)p_k + (wl^i_\omega - c^i_\omega)\omega_k - \frac{1}{1+r}(z^i_p p_k + z^i_\omega \omega_k) \right]. \tag{A.2f}$$

For the derivation of these difference equations, the same conditions have been used as for the derivation of the first-order conditions in eqs. (A.1).

We assume that there exists an optimum policy and that it converges to a steady state.[7] We assume also that the tax parameters are constant over periods. Moreover, in the steady state all the variables involved are constant over periods. By substituting the steady-state value of W_j in the difference

[7]Neither the existence nor the convergence of an optimum path is proved.

equations, eqs. (A.2), into the steady-state version of the first-order conditions, eqs. (A.1), and using Roy's identities[8] and the partial derivatives of the steady-state version of the individual lifetime budget constraint, eq. (2), in the steady state we obtain the following homogeneous-equation system:

$$\sum_i A_1^i + \sum_i A_2^i(1+\tau)q_t - \sum_i A_3^i \omega_t = 0, \tag{A.3a}$$

$$\sum_i A_2^i q + \sum_i A_2^i(1+\tau)q_\tau - \sum_i A_3^i \omega_\tau = 0, \tag{A.3b}$$

$$\sum_i A_2^i(1+\tau)q_{t^r} - \sum_i A_3^i \omega_{t^r} = 0, \tag{A.3c}$$

$$\sum_i A_2^i(1+\tau)q_{t^w} - \sum_i A_3^i \omega_{t^w} = 0, \tag{A.3d}$$

$$\sum_i A_2^i(1+\tau)q_m - \sum_i A_3^i \omega_m - \sum_i A_4^i = 0, \tag{A.3e}$$

$$\phi N_s[(1+n)(1+\rho)-(1+r)] - \sum_i A_2^i(1+\tau)q_k + \sum_i A_3^i \omega_k = 0. \tag{A.3f}$$

The only solution to this homogeneous-equation system is

$$\sum_i A_1^i \equiv \sum_i [\alpha^i c^i - \phi(c^i + tc_t^i + \kappa z_t^i + t^w w l_t^i)] = 0, \tag{A.4a}$$

$$\sum_i A_2^i \equiv \sum_i [\alpha^i z^i - \phi(z^i + tc_p^i + \kappa z_p^i + t^w w l_p^i)] = 0, \tag{A.4b}$$

$$\sum_i A_3^i \equiv \sum_i [\alpha^i l^i - \phi(l^i - tc_\omega^i - \kappa z_\omega^i - t^w w l_\omega^i)] = 0, \tag{A.4c}$$

$$\sum_i A_4^i \equiv \sum_i [\alpha^i - \phi(1 - tc_m^i - \kappa z_m^i - t^w w l_m^i)] = 0, \tag{A.4d}$$

where $\kappa = [p - 1/(1+n)(1+\rho)]$, and $\alpha^i \equiv \beta^i \lambda^i \equiv (\partial\Omega/\partial v^i)(\partial v^i/\partial m)$ is the social marginal utility of income accruing to individual i, with $\lambda^i \equiv (\partial v^i/\partial m)$ being the marginal private utility of income of individual i. This solution in turn requires from eq. (A.3f) that

[8]For example, $\mathrm{d}v^i/\mathrm{d}t = -\lambda^i[c^i + z^i(1+\tau)q_t - l^i\omega_t]$. The first term in the bracket is a direct effect, while the last two terms are indirect effects via the changes in the gross interest rate and the wage rate, which we call general equilibrium effects.

$$1+r=(1+n)(1+\rho). \tag{A.5}$$

Eq. (A.5) is often referred to as the *modified golden rule*. This result implies that the capital stock per capita is set at its first-best level. If $\rho=0$, this yields the golden rule itself with $r=n$. By substituting eq. (A.5) into the definition of κ, we can rewrite

$$\kappa\equiv p-\frac{1}{(1+n)(1+\rho)}=p-\frac{1}{1+r}=\left(\tau+\frac{t^{\mathrm{r}}r}{1+r}\right)[1+(1-t^{\mathrm{r}})r]^{-1}.$$

Thus, in the steady state, κ is interpreted as the post-tax and pre-tax price difference of second-period consumption.

Using the assumption of $\alpha=\phi=\lambda$,[9] and the definition of covariance, we can rewrite eqs. (A.4) as follows:

$$\operatorname{cov}(\lambda^i/\lambda, c^i)=tc_t+\kappa z_t+t^{\mathrm{w}}wl_t, \tag{A.6a}$$

$$\operatorname{cov}(\lambda^i/\lambda, z^i)=tc_p+\kappa z_p+t^{\mathrm{w}}wl_p, \tag{A.6b}$$

$$\operatorname{cov}(\lambda^i/\lambda, l^i)=-(tc_\omega+\kappa z_\omega+t^{\mathrm{w}}wl_\omega), \tag{A.6c}$$

$$tc_m+\kappa z_m+t^{\mathrm{w}}wl_m=0. \tag{A.6d}$$

Note that the terms c_t, z_t, and l_t denote the direct marginal effect of a change in the rate of the first-period consumption tax, not the total marginal effect.

Using the properties of the Slutsky equation, eqs. (A.6) can be simplified in matrix form:

$$\begin{bmatrix} S_{ct} & S_{zt} & S_{lt} \\ S_{cp} & S_{zp} & S_{lp} \\ -S_{c\omega} & -S_{z\omega} & -S_{l\omega} \end{bmatrix}\begin{bmatrix} t \\ \kappa \\ t^{\mathrm{w}}w \end{bmatrix}=\begin{bmatrix} \operatorname{cov}(\lambda^i/\lambda, c^i) \\ \operatorname{cov}(\lambda^i/\lambda, z^i) \\ \operatorname{cov}(\lambda^i/\lambda, l^i) \end{bmatrix}, \tag{A.7}$$

where S_{ct}, S_{zp}, $S_{l\omega}$, etc. are the Slutsky terms corresponding to the Marshallian versions of c_t, z_p, l_ω, etc. respectively. The coefficient matrix of (A.7) is a matrix of substitution terms which is negative semidefinite. These Slutsky terms show the following properties:

$$S_{cp}=S_{zt}, \quad S_{c\omega}=-S_{lt}, \quad S_{lp}=-S_{z\omega}.$$

Define compensated elasticities as

[9]For interpretations of these assumptions, see eq. (15).

$$\varepsilon_{ct} \equiv [(1+t)/c]S_{ct}, \quad \varepsilon_{cp} \equiv (p/c)S_{cp}, \quad \varepsilon_{c\omega} \equiv (\omega/c)S_{c\omega},$$

$$\varepsilon_{zt} \equiv [(1+t)/z]S_{zt}, \quad \varepsilon_{zp} \equiv (p/z)S_{zp}, \quad \varepsilon_{z\omega} \equiv (\omega/z)S_{z\omega},$$

$$\varepsilon_{lt} \equiv [(1+t)/l]S_{lt}, \quad \varepsilon_{lp} \equiv (p/l)S_{lp}, \quad \varepsilon_{l\omega} \equiv (\omega/l)S_{l\omega}.$$

Using the properties of Slutsky terms and compensated elasticities, we can rewrite matrix equation (A.7) as follows:

$$\begin{bmatrix} \varepsilon_{ct} & \varepsilon_{cp} & -\varepsilon_{c\omega} \\ \varepsilon_{zt} & \varepsilon_{zp} & -\varepsilon_{z\omega} \\ \varepsilon_{lt} & \varepsilon_{lp} & -\varepsilon_{l\omega} \end{bmatrix} \begin{bmatrix} t/(1+t) \\ \kappa/p \\ t^{\mathrm{w}}/(1-t^{\mathrm{w}}) \end{bmatrix} = \begin{bmatrix} \mathrm{cov}_c \\ \mathrm{cov}_z \\ \mathrm{cov}_l \end{bmatrix}, \tag{A.8}$$

where $\mathrm{cov}_c \equiv \mathrm{cov}(\lambda^i/\lambda, c^i/c)$ is a normalized covariance which characterizes the relation between the social marginal utility of income and the first-period consumption volume, and similarly for cov_z and cov_l. $t/(1+t)$ represents the proportion of consumption taxes to the total expenditure on the first-period consumption good, and similarly for κ/p and $t^{\mathrm{w}}/(1-t^{\mathrm{w}})$. The greater these proportions, the higher the rate of taxes.

We assume that all the compensated elasticities are constant in the neighborhood of the optimum point. For there to be a unique solution we require that the coefficient matrix of (A.8) be non-singular, i.e. that it be negative definite. Then the solutions are

$$\frac{t}{1+t} = \frac{D_1}{D}, \quad \frac{\kappa}{p} = \frac{D_2}{D}, \quad \frac{t^{\mathrm{w}}}{1-t^{\mathrm{w}}} = \frac{D_3}{D}, \tag{A.9}$$

where D denotes the determinant of the coefficient matrix of (A.8). D_1 denotes the determinant of the coefficient matrix with the first column replaced by the covariance matrix on the right-hand side of (A.8), and similarly for D_2 and D_3. These proportions are determined by different sums of products of the compensated own and cross-elasticities and the covariances.

With the assumption of $\tau \equiv t$, the explicit solutions are

$$t = \frac{D_1}{D - D_1}, \tag{A.10a}$$

$$t^{\mathrm{r}} = \frac{(1+n)(1+\rho)}{\{(1+n)(1+\rho)-1\}} \frac{(D_2 - D_1)}{D - D_1}, \tag{A.10b}$$

$$t^{\mathrm{w}} = \frac{D_3}{D + D_3}. \tag{A.10c}$$

To interpret this result we need information about such parameters as the compensated elasticities and covariances. To estimate these parameters is an empirical matter. The revenues from these non-lump-sum taxes finance the lump-sum transfer to the present generation and the principal and interest payments on the bonds. The distortions in the economy owing to these taxes are the opportunity costs of the distributional objectives of the government. In other words, these distortions represent a kind of trade-off between efficiency and equity.

One interesting result of these solutions concerns the rate of interest-income tax, t^r. It can be negative, positive or zero depending on the sizes of the terms D_2 and D_1. It is also so that, ceteris paribus, the lower the interest rate, the higher the rate of interest-income tax, and vice versa. Since the *modified golden rule* holds, this in turn implies that the lower the population growth or the time preference, the higher the rate of taxes on interest earnings, and vice versa.

As the solutions in eqs. (A.10) show, the optimal tax rates depend crucially on the determinants, which in turn depend on the compensated elasticities and covariances. We cannot, however, determine the sizes and signs of these parameters unless we have empirical information about these terms. To estimate these parameters is an empirical matter. Thus, we cannot interpret the result from this model in more exhaustive way.

References

Allen, F., 1982, Optimal linear income taxation with general equilibrium effects on wages, Journal of Public Economics 17, 135–143.

Atkinson, A.B. and A. Sandmo, 1980, Welfare implications of the taxation of savings, Economic Journal 90, 529–549.

Atkinson, A.B. and J.E. Stiglitz, 1976, The design of tax structure: Direct versus indirect taxation, Journal of Public Economics 6, 55–75.

Atkinson, A.B. and J.E. Stiglitz, 1980, Lectures on public economics (McGraw-Hill, London).

Bellman, R.E. and S.E. Dreyfus, 1962, Applied dynamic programming (Princeton University Press, Princeton, NJ).

Diamond, P.A., 1965, National debt in a neoclassical growth model, American Economic Review 55, 1126–1150.

Dupuit, J., 1844, De la mesure de l'utilité des travaux publics, Annales des Pontes et Chaussées 8. Translated and reprinted as: On the measurement of the utility of public works, in: K.J. Arrow and T. Scitovsky, eds., 1969, AEA Readings in welfare economics, 255–283.

Intriligator, M.D., 1971, Mathematical optimization and economic theory (Prentice-Hall, Englewood Cliffs, NJ).

Mirrlees, J.A., 1971, An exploration in the theory of optimum income taxation, Review of Economic Studies 38, 175–208.

Ordover, J.A., 1976, Distributive justice and optimal taxation of wages and interest in a growing economy, Journal of Public Economics 5, 139–160.

Ordover, J.A. and E.S. Phelps, 1975, Linear taxation of wealth and wages for intragenerational lifetime justice: Some steady-state cases, American Economic Review 65, 660–673.

Ordover, J.A. and E.S. Phelps, 1979, The concept of optimal taxation in the overlapping-generations model of capital and wealth, Journal of Public Economics 12, 1–26.

Persson, T., 1985, Deficits and intergenerational welfare in open economies, Journal of International Economics 19, 67–84.

Pestieau, P.M., 1974, Optimal taxation and discount rate for public investment in a growth setting, Journal of Public Economics 3, 217–235.

Ramsey, F.P., 1927, A contribution to the theory of taxation, Economic Journal 37, 47–61.
Samuelson, P.A., 1958, An exact consumption-loan model of interest with or without the social contrivance of money, Journal of Political Economy 66, 467–482.
Sandmo, A., 1976, Optimal taxation: An introduction to the literature, Journal of Public Economics 6, 37–54.
Sheshinski, E., 1972, The optimal linear income tax, Review of Economic Studies 39, 297–302.
Stern, N.H., 1984, Optimum taxation and tax policy, International Monetary Fund Staff Papers 31, June, 339–378.
Svensson, L.-G. and J.W. Weibull, 1987, Constrained Pareto-optimal taxation of labour and capital incomes, Journal of Public Economics 34, 355–366.
U.S. Bureau of the Census, 1987, Statistical abstract of the United States: 1988, 108th edn. (Washington, DC).