Chapter I.3

APPLICATIONS OF PROFIT FUNCTIONS

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1. The Profit Function – An Alternative Derivation

1.1. *Introduction*

In a pioneering attempt, McFadden (1966) extends the concept of cost functions to revenue functions and profit functions and proves for the first time the McFadden Duality Theorem – the profit function analog of the Shephard (1953) – Uzawa (1964) Duality Theorem on cost and production functions. The purpose of this chapter is to provide an alternative derivation under conditions which guarantee twice differentiability of both the production function and the corresponding dual profit function, to characterize equivalent structural properties of the production function and the profit function, and to propose a variety of econometric applications of the profit function.

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Let

$$Y = F(X_1,\dots,X_m;Z_1,\dots,Z_n)$$

be the production function of a firm, where the X_i's and the Z_j's are the variable and the fixed inputs respectively. Then short-run profit, defined as revenue less variable costs, is given by

$$\begin{aligned} P &= pF(\mathbf{X},\mathbf{Z}) - \sum_{i=1}^{m} q_i^* X_i \\ &= p\left[F(\mathbf{X},\mathbf{Z}) - \sum_{i=1}^{m} q_i X_i\right] \\ &= p\,[F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}] \end{aligned}$$

where

p = nominal (money) price of output,
q_i^* = nominal price of input i,
$q_i = q_i^*/p$, normalized price of input i,

and $\mathbf{X}$, $\mathbf{Z}$ and $\mathbf{q}$ are the vectors of X_i's, Z_j's and q_i's, respectively.

It is assumed that the objective of productive activity is the maximization of short-run profit and that the firm is a price-taker in the output and variable inputs markets. Thus, the firm maximizes profit with respect to $\mathbf{X}$ taking p, $\mathbf{q}^*$ and $\mathbf{Z}$ as given. The *profit function* Π is a function of p, $\mathbf{q}^*$ and $\mathbf{Z}$ which gives for each set of values p, $\mathbf{q}^*$, $\mathbf{Z}$ the *maximized* value of profit

$$\Pi(p,\mathbf{q}^*;\mathbf{Z}) = p\,[F(\mathbf{X}^*,\mathbf{Z}) - \mathbf{q}'\mathbf{X}^*],$$

where the X_i^*'s are the optimized quantities of the variable inputs.

Before proceeding further, one may observe that maximization of profit is equivalent to the maximization of normalized profit, P^*,[1] defined by

$$P^* = P/p = F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X},$$

so that the X_i^*'s are identical for the two problems. It is clear that the corresponding normalized profit function is given by

$$\begin{aligned} \Pi^* &= F(\mathbf{X}^*,\mathbf{Z}) - \mathbf{q}'\mathbf{X}^* \\ &= G(\mathbf{q},\mathbf{Z}). \end{aligned}$$

The normalized profit function $G(\mathbf{q},\mathbf{Z})$ is more convenient to work with

[1] This was referred to as the "Unit-Output-Price" or "UOP" profit in Lau (1969c). The terminology "normalized profit" is due to Jorgenson and Lau (1974a and 1974b).

for the purposes at hand but the one-to-one correspondence between $\Pi(p,\mathbf{q}^*,\mathbf{Z})$ and $G(\mathbf{q},\mathbf{Z})$ should be obvious.

1.2. *Properties of the Production Function*

The production function is assumed to have certain properties. Let $\bar{R}^n_+$ and $\bar{R}^m_+$ denote the closed non-negative orthants of R^n and R^m, and R^m_+ the interior of the non-negative orthant of R^m. The assumptions on the production function are as follows:

(F.1) *Domain.* F is a finite, non-negative, real-valued function defined on $\bar{R}^m_+ \times \bar{R}^n_+$. For each $\mathbf{Z} \in \bar{R}^n_+$, $F(\mathbf{0},\mathbf{Z}) = 0$.

(F.2) *Continuity.* F is continuous on $\bar{R}^m_+ \times \bar{R}^n_+$.

(F.3) *Smoothness.* For each $\mathbf{Z} \in \bar{R}^n_+$, F is continuously differentiable on R^m_+, and the Euclidean norm of the gradient of F with respect to $\mathbf{X}$ is unbounded for any sequence of $\mathbf{X}$ in R^m_+ converging to a boundary point of $\bar{R}^m_+$. For each $\mathbf{X} \in \bar{R}^m_+$, F is continuously differentiable on R^n_+.

For each $\mathbf{Z} \in \bar{R}^n_+$, the gradient of F with respect to $\mathbf{X}$ on R^m_+ will be denoted $\nabla_X F(\mathbf{X},\mathbf{Z})$.

(F.4) *Monotonicity.* F is non-decreasing in $\mathbf{X}$ and $\mathbf{Z}$ on $\bar{R}^m_+ \times \bar{R}^n_+$ and strictly increasing in $\mathbf{X}$ and $\mathbf{Z}$ on $R^m_+ \times R^n_+$.

(F.5) *Concavity.* For each $\mathbf{Z} \in \bar{R}^n_+$, F is concave on $\bar{R}^m_+$ and locally strongly concave on R^m_+.

Definition. A function is *strongly concave* on a convex set C if there exists $\delta > 0$ such that[2]

$$F((1-\lambda)\mathbf{X}_1 + \lambda\mathbf{X}_2) \geqq (1-\lambda)F(\mathbf{X}_1) + \lambda F(\mathbf{X}_2) + \lambda(1-\lambda)\delta(\mathbf{X}_1 - \mathbf{X}_2)'(\mathbf{X}_1 - \mathbf{X}_2), \quad 0 \leqq \lambda \leqq 1, \quad \forall \mathbf{X}_1, \mathbf{X}_2 \in C.$$

An example of a strongly concave function is $F(X) = -X^2$. A function is *locally strongly concave* if there exists such a δ for every proper convex subset of C. An example of a locally strongly concave (but not strongly

[2]See Roberts and Varberg (1973, p. 268).

concave) function on $\bar{R}_+$ is $F(X) = 1 - e^{-X}$.[3] Obviously local strong concavity implies strict concavity.

> (F.6) *Twice Differentiability.* For each $\mathbf{Z} \in \bar{R}^n_+$, F is twice continuously differentiable on R^m_+.

The concavity and twice differentiability assumptions together imply that for each $\mathbf{Z} \in \bar{R}^n_+$ the Hessian matrix of F with respect to $\mathbf{X}$ is negative definite on R^m_+.

> (F.7) *Boundedness.* For each $\mathbf{Z} \in \bar{R}^n_+$,
>
> $$\lim_{\lambda \to \infty} \frac{F(\lambda \mathbf{X}, \mathbf{Z})}{\lambda} = 0 \qquad \forall \mathbf{X} \in \bar{R}^m_+.$$

The boundedness assumption ensures that a bounded and attainable solution exists for the normalized profit maximization problem for all $\mathbf{q} \in R^m_+$. This assumption is sufficient even if the production function is not differentiable, that is, in the absence of (F.3) and (F.6). For the purpose at hand, one may have adopted for (F.7) the alternative assumption that for each $\mathbf{Z} \in \bar{R}^n_+$, the range of $\nabla_X F(\mathbf{X},\mathbf{Z})$ is all of R^m_+.

An example of a function for which (F.7) fails is $F(X) = X + X^{1/2}$. For this production function there does not exist a profit maximum if $q \leqq 1$. An example of a function which satisfies (F.7), but fails (F.3) is $F(X) = 1 - e^{-X}$.

Assumptions (F.1) through (F.6) are sufficient to ensure that, if a solution $\mathbf{X}^*$ to the normalized profit maximization problem exists for a given $\mathbf{q}$ and $\mathbf{Z}$, the solution will be unique and lies in R^m_+. The additional Assumption (F.7) is needed to ensure that such a solution exists for arbitrary $\mathbf{q} \in R^m_+$ and $\mathbf{Z} \in \bar{R}^n_+$. We therefore have the following two lemmas:

> *Lemma I*-1. Under Assumptions (F.1) through (F.6), for each $\mathbf{q} \in R^m_+$, $\mathbf{Z} \in \bar{R}^n_+$, if a vector $\mathbf{X}^*$ exists such that
>
> $$F(\mathbf{X}^*,\mathbf{Z}) - \sum_{i=1}^{m} q_i X_i^* \geqq F(\mathbf{X},\mathbf{Z}) - \sum_{i=1}^{m} q_i X_i, \qquad \forall \mathbf{X} \in \bar{R}^m_+,$$
>
> then $\mathbf{X}^*$ is unique and lies in R^m_+.

[3]Under the additional assumption of twice differentiability, local strong concavity implies negative definiteness of the Hessian matrix. Compare the concept of differential strict quasi-concavity which implies that the Hessian matrix is negative semi-definite with rank $(m - 1)$. See Chapter I.1.

Proof: The Kuhn–Tucker necessary condition for a maximum implies that

$$\nabla_X F(\mathbf{X}^*,\mathbf{Z}) \leqq \mathbf{q},$$

with equality in each component of $\nabla_X F$ for which the corresponding component of $\mathbf{X}^*$ is positive. However, if any component of $\mathbf{X}^*$ is zero, by (F.3) and (F.4) $\nabla_X F$ is unbounded and positive, thus violating the Kuhn–Tucker condition. Hence $\mathbf{X}^*$ must be positive and lies in R^m_+. Finally, by (F.5) $\mathbf{X}^*$ must be unique. Q.E.D.

Lemma I-2. Under Assumptions (F.1) through (F.7), for each $\mathbf{q} \in R^m_+$, $\mathbf{Z} \in \bar{R}^n_+$, there exists a unique vector

$$\mathbf{X}^* = \mathbf{X}^*(\mathbf{q},\mathbf{Z}) \in R^m_+$$

such that

$$F(\mathbf{X}^*,\mathbf{Z}) - \mathbf{q}'\mathbf{X}^* \geqq F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}, \qquad \forall \mathbf{X} \in \bar{R}^m_+.$$

Further, the function $\mathbf{X}^*(\mathbf{q},\mathbf{Z}): R^m_+ \times \bar{R}^n_+ \to R^m_+$ is continuous on R^m_+ for each $\mathbf{Z} \in \bar{R}^n_+$ and continuous on $\bar{R}^n_+$ for each $\mathbf{q} \in R^m_+$. For each $\mathbf{Z} \in \bar{R}^m_+$, $\mathbf{X}^*$ is continuously differentiable on R^m_+.[4]

Proof: Under Assumptions (F.1) through (F.6), for each $\mathbf{Z} \in \bar{R}^n_+$, normalized profit, $P^* = F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}$, is a closed, proper concave function in $\mathbf{X}$ on $\bar{R}^m_+$ for all $\mathbf{q} \in R^m_+$.[5] For a given $\mathbf{Z}$ and $\mathbf{q}$, a finite and attainable maximum exists for this closed, proper, concave function if and only if the function P^* has no directions of recession in $\mathbf{X}$. [See Rockafellar (1970, Theorem 27.1, pp. 264–265; and also Theorem 13.3 and its corollaries, pp. 116–117).] The directions of recession of P^* are the vectors $\mathbf{y} \neq 0$, $\mathbf{y} \in \operatorname{dom} P^*$ (domain of P^*), such that

$$\lim_{\lambda \to \infty} P^* \frac{(\lambda \mathbf{y},\mathbf{Z})}{\lambda} \geqq 0.^6$$

Thus, in order for P^* to have no directions of recession one must have

$$\lim_{\lambda \to \infty} \frac{F(\lambda \mathbf{X},\mathbf{Z})}{\lambda} - \mathbf{q}'\mathbf{X} < 0, \qquad \forall \mathbf{X} \in \operatorname{dom} F, \quad \mathbf{X} \neq 0.$$

Since $q \in R^m_+$, $\operatorname{dom} F = \bar{R}^m_+$, $q'X > 0$. Thus, one concludes that

[4] See Appendix A.3, Lemma 15.4.

[5] A concave extended-real-valued function is proper if it nowhere takes the value of ∞ and is finite for at least one value of its arguments.

[6] See Rockafellar (1970, pp. 66–67, and especially Theorem 8.5 and its corollaries).

$\lim_{\lambda\to\infty} F(\lambda\mathbf{X},\mathbf{Z})/\lambda \leqq 0$ is necessary, But $F(\lambda\mathbf{X},\mathbf{Z}) \geqq 0$, by (F.1), thus $\lim_{\lambda\to\infty} F(\lambda\mathbf{X},\mathbf{Z})/\lambda = 0$ is necessary and sufficient to ensure that no direction of recession exists. Hence, with (F.7) a finite and attainable solution exists. But this argument works for all $\mathbf{q} \in R^m_+$. Thus, for all $\mathbf{q} \in R^m_+$, a finite and attainable solution exists.[7] By Lemma I-1, the optimal solution $\mathbf{X}^*$ is positive and unique.

Continuity properties of $\mathbf{X}^*$ follow from the continuity of $\nabla_X F$ on R^m_+ for each $\mathbf{Z} \in \bar{R}^n_+$ and on $\bar{R}^n_+$ for every $\mathbf{X} \in R^m_+$.

Implicit differentiation using the implicit function theorem guarantees the differentiability property of $\mathbf{X}^*$. The assumption of non-singularity of the Jacobian matrix so crucial in the application of the implicit function theorem is implied by the negative definiteness of the Hessian matrix of F with respect to $\mathbf{X}$. Q.E.D.

> *Corollary* 2.1. The normalized profit function $G(\mathbf{q},\mathbf{Z}) = F(\mathbf{X}^*(\mathbf{q},\mathbf{Z}),\mathbf{Z}) - \mathbf{q}'\mathbf{X}^*(\mathbf{q},\mathbf{Z})$ is continuous on $R^m_+ \times \bar{R}^n_+$, is twice continuously differentiable on R^m_+ for each $\mathbf{Z} \in \bar{R}^n_+$, and is continuously differentiable on R^n_+ for each $\mathbf{q} \in R^m_+$. $Y^*(\mathbf{q},\mathbf{Z}) = F(\mathbf{X}^*(\mathbf{q},\mathbf{Z}),\mathbf{Z})$ is continuous on $R^m_+ \times \bar{R}^n_+$ and is continuously differentiable on R^m_+ for each $\mathbf{Z} \in \bar{R}^n_+$.

Proof: The proof follows from repeated application of the chain rule for partial differentiation and the fact that

$$\frac{\partial F}{\partial \mathbf{X}}(\mathbf{X}^*,\mathbf{Z}) - \mathbf{q} = 0 \quad \text{Q.E.D.}$$

1.3. *Duality*

The duality between production functions and normalized profit functions has been established under rather general circumstances in Chapter I.1 and elsewhere.[8] Our purpose here is to establish properties of the class of normalized profit functions which correspond to the class of production functions which satisfies our Assumptions (F.1) through (F.7) and to demonstrate that there exists a one-to-one correspondence

[7]Note that this establishes the domain of G as all of R^m_+. See Chapter I.1.

[8]See also Cass (1974), Diewert (1973a and 1974a), Jorgenson and Lau (1974a and 1974b), and Lau (1976a). Jorgenson and Lau base their duality results on the conjugacy correspondence of closed, proper convex functions.

between the members of the two classes. For every production function which satisfies Assumptions (F.1) through (F.7) one can define a normalized profit function $G(\mathbf{q},\mathbf{Z})$ on $R_+^m \times \bar{R}_+^n$

$$G(\mathbf{q},\mathbf{Z}) = \sup_{\mathbf{X}}\{F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}\}.$$

For each $\mathbf{Z} \in \bar{R}_+^n$, $G(\mathbf{q},\mathbf{Z})$ is also referred to as the conjugate of $F(\mathbf{X},\mathbf{Z})$. By Lemma I-2, a finite and attainable maximum always exists for $q \in R_+^m$. Thus,

$$G(\mathbf{q},\mathbf{Z}) = \max_{\mathbf{X}}\{F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}\}.$$

It will be shown that the normalized profit function $G(\mathbf{q},\mathbf{Z})$ corresponding to a production function satisfying Assumptions (F.1) through (F.7) possesses the following properties:

(G.1) *Domain.* G is a finite, positive, real-valued function defined on $R_+^m \times \bar{R}_+^n$.

(G.2) *Continuity.* G is continuous on $R_+^m \times \bar{R}_+^n$.

(G.3) *Smoothness.* For each $\mathbf{Z} \in \bar{R}_+^n$, G is continuously differentiable on R_+^m, and the Euclidean norm of the gradient of G with respect to $\mathbf{q}$ is unbounded for any sequence of $\mathbf{q}$ in R_+^m converging to a boundary point of $\bar{R}_+^m$. For each $\mathbf{q} \in R_+^m$, F is continuously differentiable on R_+^n.

(G.4) *Monotonicity.* $G(\mathbf{q},\mathbf{Z})$ is non-increasing $\mathbf{q}$ and non-decreasing in $\mathbf{Z}$ on $R_+^m \times \bar{R}_+^n$ and strictly decreasing in $\mathbf{q}$ and strictly increasing in $\mathbf{Z}$ on $R_+^m \times R_+^n$.

(G.5) *Convexity.* For each $\mathbf{Z} \in \bar{R}_+^n$, $G(\mathbf{q},\mathbf{Z})$ is locally strongly convex on R_+^m.

Definition. A function F is locally strongly convex if $-F$ is locally strongly concave.

(G.6) *Twice Differentiability.* For each $\mathbf{Z} \in \bar{R}_+^n$, $G(\mathbf{q},\mathbf{Z})$ is twice continuously differentiable on R_+^m.

The convexity and twice differentiability assumptions together imply that for each $\mathbf{Z} \in \bar{R}^n_+$ the Hessian matrix of G with respect to $\mathbf{q}$ is positive definite on R^m_+.

(G.7) *Boundedness.* For each $\mathbf{Z} \in \bar{R}^n_+$,

$$\lim_{\lambda \to \infty} \frac{G(\lambda \mathbf{q},\mathbf{Z})}{\lambda} = 0, \qquad \forall \mathbf{q} \in R^m_+.$$

Lemma I-3. Under Assumptions (F.1) through (F.7), the normalized profit function satisfies Assumptions (G.1) through (G.7).

Proof:
(G.1) From Lemma I-2, $G(\mathbf{q},\mathbf{Z})$ is a finite and real-valued function defined on $R^m_+ \times \bar{R}^n_+$ since a finite and attainable maximum exists. And because for each $\mathbf{Z} \in \bar{R}^n_+$, $F(\mathbf{0},\mathbf{Z}) = 0$, $G(\mathbf{q},\mathbf{Z}) \geqq 0$. If $G(\mathbf{q},\mathbf{Z}) = 0$ for any $\mathbf{q} \in R^m_+$, then a profit-maximizing vector $\mathbf{X}^*$ is $\mathbf{X}^* = 0$. However, this contradicts Lemma I-2, which states that $\mathbf{X}^* \in R^m_+$. Thus $G(\mathbf{q},\mathbf{Z})$ is positive.

(G.2) Since $G(\mathbf{q},\mathbf{Z}) = \max_{\mathbf{X}}\{F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}\}$ it follows that for each $\mathbf{q} \in R^m_+$, $G(\mathbf{q},\mathbf{Z})$ is continuous in $\mathbf{Z}$ on $\bar{R}^n_+$ by (F.2). In addition, for each $\mathbf{Z} \in \bar{R}^n_+$, $G(\mathbf{q},\mathbf{Z})$ is convex on R^m_+ [proved under (G.5) below]. Thus, by a theorem in Rockafellar (1970, Theorem 10.7, pp. 89–90), $G(\mathbf{q},\mathbf{Z})$ is continuous on $R^m_+ \times \bar{R}^n_+$.

(G.3) Smoothness in $\mathbf{q}$ is implied by (F.5) [see Rockafellar (1970, Theorem 26.3, pp. 253–254)]. Differentiability in $\mathbf{Z}$ follows from the fact that $F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}$ is continuously differentiable in $\mathbf{Z}$ on R^n_+ and that $G(\mathbf{q},\mathbf{Z}) = \max_{\mathbf{X}}\{F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}\}$.

(G.4) Let $\mathbf{X}^*_1$ and $\mathbf{X}^*_2 \in R^m_+$ be the profit-maximizing inputs at $\mathbf{q}_1$ and $\mathbf{q}_2$, respectively. Then,

$$G(\mathbf{q}_1,\mathbf{Z}) > F(\mathbf{X}^*_2,\mathbf{Z}) - \mathbf{q}'_1\mathbf{X}^*_2,$$
$$G(\mathbf{q}_2,\mathbf{Z}) > F(\mathbf{X}^*_1,\mathbf{Z}) - \mathbf{q}'_2\mathbf{X}^*_1.$$

Suppose $\mathbf{q}_1$ is strictly greater than $\mathbf{q}_2$ (in at least one component) then $F(\mathbf{X}^*_1,\mathbf{Z}) - \mathbf{q}'_2\mathbf{X}^*_1 > F(\mathbf{X}^*_1,\mathbf{Z}) - \mathbf{q}'_1\mathbf{X}^*_1 = G(\mathbf{q}_1,\mathbf{Z})$. Hence $G(\mathbf{q}_2,\mathbf{Z}) > G(\mathbf{q}_1,\mathbf{Z})$. Monotonicity in $\mathbf{Z}$ follows from the fact that $G(\mathbf{q},\mathbf{Z}) = \max_{\mathbf{X}}\{F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}\}$.

(G.5) Local strong convexity is implied by (F.3) and (F.5).

(G.6) Twice differentiability is implied by (F.5) and (F.6) through the chain rule.

(G.7) Boundedness follows from the fact that for each $\mathbf{Z} \in \bar{R}^n_+$, the domain of $F(\mathbf{X},\mathbf{Z})$ is $\bar{R}^m_+$, the support function of which is given by

$$\begin{aligned} \delta^*(\mathbf{X}^*|\bar{R}^m_+) &= 0, \quad && \mathbf{X}^* \in \bar{R}^m_-, \\ &= +\infty, && \mathbf{X}^* \notin \bar{R}^m_-. \end{aligned}$$

But this is also the recession function of the conjugate of $F(\mathbf{X},\mathbf{Z})$, $G(\mathbf{q},\mathbf{Z})$, with $\mathbf{q}$ identified with $-\mathbf{X}^*$ [see Rockafellar (1970, Theorem 13.3, p. 116)]. The recession function of $G(\mathbf{q},\mathbf{Z})$ is given by

$$\lim_{\lambda \to \infty} \frac{G(\lambda \mathbf{q},\mathbf{Z})}{\lambda}, \qquad \mathbf{q} \in R^m_+.$$

Thus, one has

$$\lim_{\lambda \to \infty} \frac{G(\lambda \mathbf{q},\mathbf{Z})}{\lambda} = 0, \qquad \mathbf{q} \in R^m_+. \quad \text{Q.E.D.}$$

Given a normalized profit function $G(\mathbf{q},\mathbf{Z})$ one may define its conjugate as

$$F^*(\mathbf{X},\mathbf{Z}) = \inf_{\mathbf{q}} \{G(\mathbf{q},\mathbf{Z}) + \mathbf{q}'\mathbf{X}\}.$$

Under Assumptions (G.1) through (G.7), $G(\mathbf{q},\mathbf{Z})$ is a closed proper convex function on $R^m_+ \times \bar{R}^n_+$, hence its conjugate function is unique and equal to $F(\mathbf{X},\mathbf{Z})$ itself. [For the one-to-one correspondence between closed proper convex functions and its conjugate, see Chapter I.1 and Rockafellar (1970, ch. 12).] Hence all one needs to do is to verify that $F^*(\mathbf{X},\mathbf{Z})$ in fact satisfies Assumption (F.1) through (F.7). Thus one has:

Lemma I-4. Under Assumptions (G.1) through (G.7), the production function satisfies Assumptions (F.1) through (F.7).

Proof: The proof parallels the proof of Lemma I-3. The only exception is that of continuity of $F(\mathbf{X},\mathbf{Z})$ on the boundary of R^m_+. This follows from the fact that $F(\mathbf{X},\mathbf{Z}) = \inf_{\mathbf{q}}\{G(\mathbf{q},\mathbf{Z}) + \mathbf{q}'\mathbf{X}\}$ is a closed proper concave function and bounded below on every bounded subset of R^m_+. Hence, $F(\mathbf{X},\mathbf{Z})$ may be uniquely extended to a continuous finite concave function on $\bar{R}^m_+$. [See Rockafellar (1970, pp. 84–86) and also Lemma 12.7

in Appendix A.3 of this volume.] It is then possible to set $F(\mathbf{0},\mathbf{Z}) = 0$. Q.E.D.

We conclude this section by noting that it is possible to relax the assumption that the domain of $F(\mathbf{X},\mathbf{Z})$ is all of $\bar{R}^m_+$, or that the range of $\nabla_X F$ is all of R^m_+ as is done in Chapter I.1 and Jorgenson and Lau (1974a and 1974b). It is also possible to relax the assumption that $|\nabla_X F|$ becomes unbounded as X approaches the boundary of its domain from the interior, requiring only that the range of $\nabla_X F$ on the domain of F has a non-empty interior. Under these mild modifications, the properties of continuity, differentiability, monotonicity, concavity and twice differentiability still imply corresponding properties on the dual, only that the domains of definition are now a pair of open convex sets C and C^*, such that $C \subset \text{int}(\text{dom } F)$ and $C^* \subset \text{int } D$ where D is the range of $\nabla_X F$ on C.[9]

1.4. *The Legendre Transformation*

One way of obtaining the normalized profit function is to solve the maximization problem first for the derived demand functions and then substitute these back into the formula for normalized profit given by

$$P^* = [F(\mathbf{X},\mathbf{Z}) - \mathbf{q}'\mathbf{X}].$$

The difficulty with this method is that only for relatively simple production functions can one solve the profit-maximization problem explicitly to obtain closed form solutions for the derived demand functions. An alternative method for constructing the normalized profit function and for studying the behavior of the normalized profit function (without actually constructing it), based on the classical Legendre transformation,[10] will be given below.

The Legendre transformation is a change of variables of a function from point coordinates to plane coordinates. It is based partly on the notion that a system of partial differential equations may be used to define two or more sets of functions through transformation of variables. In the present case, it can be shown that the production function

[9]For further discussion of this point, see Rockafellar (1970, pp. 251–260).

[10]A succinct exposition of the Legendre transformation may be found in Lanczos (1966, Ch. VI). See also Courant and Hilbert (1953, Vol. II, Ch. I, pp. 32–39). For a discussion from a more modern point of view, see Rockafellar (1970).

and the normalized profit function are connected by the Legendre transformation.

Consider a given function of m variables V_i's and n parameters p_i's,

$$f = (V_1,\dots,V_m;p_1,\dots,p_n),$$

new variable T_i's may be introduced by means of the following transformation:

$$T_i = \frac{\partial f}{\partial V_i}, \qquad i = 1,\dots,m, \tag{I-1}$$

which is called the Legendre transformation. The variables V_i's are replaced by the variables T_i's. f is assumed to be locally strongly concave in the V_i's so that the transformation is non-singular and hence invertible. Thus equation (I-1) may be solved, expressing the V_i's in terms of the T_i's and p_i's,

$$V_i = h_i(T_1,\dots,T_m;p_1,\dots,p_n), \qquad i = 1,\dots,m.$$

A new function g may be defined as follows:

$$g(T_1,\dots,T_m;p_1,\dots,p_n) = \sum_{i=1}^{m} h_i(\mathbf{T},\mathbf{p})T_i - f(h_1(\mathbf{T},\mathbf{p}),\dots,h_m(\mathbf{T},\mathbf{p});\mathbf{p}).$$

The function g is known as the Legendre's dual transformation of the primal function f.

Observe that

$$\frac{\partial g}{\partial T_i} = \sum_{j=1}^{m} \frac{\partial h_j}{\partial T_i} T_j + h_i - \sum_{j=1}^{m} \frac{\partial f}{\partial V_j}\cdot\frac{\partial h_j}{\partial T_i}, \qquad i = 1,\dots,m.$$

But by equation (I-1), $T_j = \partial f/\partial V_j$. Thus,

$$\begin{aligned}\frac{\partial g}{\partial T_i} &= h_i(\mathbf{T},\mathbf{p}) \\ &= V_i, \qquad i = 1,2\dots,m. \end{aligned} \tag{I-2}$$

Equation (I-2) is the inverse Legendre transformation. The variables T_i are replaced by the variables V_i's. In addition, we have

$$\begin{aligned}\frac{\partial g}{\partial p_i} &= \sum_{j=1}^{m} \frac{\partial h_j}{\partial p_i}\cdot T_j - \sum_{j=1}^{m} \frac{\partial f}{\partial V_j}\cdot\frac{\partial h_j}{\partial p_i} - \frac{\partial f}{\partial p_i}, \qquad i = 1,\dots,n, \\ &= -\frac{\partial f}{\partial p_i}, \end{aligned} \tag{I-3}$$

again by equation (I-1).

If we now compute the Legendre transformation of g, we have

$$g^*(V_1,\ldots,V_m;p_1,\ldots,p_n) = \sum_{i=1}^{m} T_i(\mathbf{V},\mathbf{p})\cdot V_i - g(T_1(\mathbf{V},\mathbf{p}),\ldots,T_m(\mathbf{V},\mathbf{p});\mathbf{p})$$
$$= f.$$

The functions f and g are linked by the following set of dual relations:

$$f(V_1,V_2,\ldots,V_m;\mathbf{p}) + g(T_1,T_2,\ldots,T_m;\mathbf{p}) = \sum_{i=1}^{m} V_i T_i,$$

$$\frac{\partial f}{\partial \mathbf{V}} = \mathbf{T}, \qquad \frac{\partial g}{\partial \mathbf{T}} = \mathbf{V}, \qquad \frac{\partial f}{\partial \mathbf{p}} + \frac{\partial g}{\partial \mathbf{p}} = 0.$$

There is also a set of transformations relating the second derivatives of f and the second derivatives of g. Starting from

$$\frac{\partial f}{\partial \mathbf{V}} = \mathbf{T},$$

one may differentiate this set of dual relations with respect to $\mathbf{T}$ obtaining

$$\left[\frac{\partial V}{\partial \mathbf{T}}\right]\left[\frac{\partial^2 f}{\partial \mathbf{V}\partial \mathbf{V}'}\right] = \left[\frac{\partial^2 g}{\partial \mathbf{T}\partial \mathbf{T}'}\right]\left[\frac{\partial^2 f}{\partial \mathbf{V}\partial \mathbf{V}'}\right] = I.$$

One may also differentiate the set of dual relations with respect to $\mathbf{p}$, obtaining

$$\left[\frac{\partial V}{\partial \mathbf{p}}\right]\left[\frac{\partial^2 f}{\partial \mathbf{V}\partial \mathbf{V}'}\right] + \left[\frac{\partial^2 f}{\partial \mathbf{p}\partial \mathbf{V}'}\right] = 0,$$

or

$$\left[\frac{\partial^2 g}{\partial \mathbf{p}\partial \mathbf{T}'}\right]\left[\frac{\partial^2 f}{\partial \mathbf{V}\partial \mathbf{V}'}\right] + \left[\frac{\partial^2 f}{\partial \mathbf{p}\partial \mathbf{V}'}\right] = 0.$$

We note that this is also a symmetric relation because

$$\left[\frac{\partial^2 g}{\partial \mathbf{T}\partial \mathbf{T}'}\right] = \left[\frac{\partial^2 f}{\partial \mathbf{V}\partial \mathbf{V}'}\right]^{-1}.$$

In terms of our problem, the production function $F(\mathbf{X},\mathbf{Z})$ may be identified as f. The normalized profit function $G(\mathbf{q},\mathbf{Z})$ may be identified as $-g$. $\mathbf{X}$ may be identified as $\mathbf{V}$. $\mathbf{Z}$ may be identified as $\mathbf{p}$. The new variables to be introduced – the plane coordinates – are set equal to

$$\mathbf{T} = \frac{\partial F}{\partial \mathbf{X}},$$

in accordance with the Legendre transformation. However, $\partial F/\partial \mathbf{X} = q$ under the assumption of profit maximization. Thus

$$\mathbf{T} = \frac{\partial F}{\partial \mathbf{X}} = \mathbf{q}.$$

and $\mathbf{q}$ may be identified as $\mathbf{T}$. The Legendre transformation may be constructed as

$$g = \sum_{i=1}^{m} T_i X_i(\mathbf{T},\mathbf{Z}) - F(X_1(\mathbf{T},\mathbf{Z}),\ldots,X_m(\mathbf{T},\mathbf{Z}),\mathbf{Z}).$$

By recognizing that $\mathbf{T} = \mathbf{q}$, we have

$$g = \sum_{i=1}^{m} q_i X_i(\mathbf{q},\mathbf{Z}) - F(X_1(\mathbf{q},\mathbf{Z}),\ldots,X_m(\mathbf{q},\mathbf{Z}),\mathbf{Z}),$$

which is precisely equal to $-G$, the negative of the normalized profit function. Moreover, from the inverse Legendre transformation

$$\frac{\partial g}{\partial \mathbf{q}} = \frac{\partial g}{\partial \mathbf{T}} = \mathbf{X},$$

$$\frac{\partial g}{\partial \mathbf{Z}} = -\frac{\partial F}{\partial \mathbf{Z}}.$$

Hence, one has

$$\frac{\partial G}{\partial \mathbf{q}} = -\mathbf{X},$$

$$\frac{\partial G}{\partial \mathbf{Z}} = \frac{\partial F}{\partial \mathbf{Z}}.$$

This set of relations is sometimes referred to as Hotelling's (1932) Lemma and is of crucial importance in applications. We may then summarize the Legendre transformation relationships between the production function and the normalized restricted profit function:

	Primal	Dual
Function:	Production function $F(\mathbf{X},\mathbf{Z})$	Normalized function $G(\mathbf{q},\mathbf{Z})$
Active variables:	$\mathbf{X}$	$\mathbf{q}$
Passive variables:	$\mathbf{Z}$	$\mathbf{Z}$

And we get the following dual transformation relations:

(1)	$F(\mathbf{X},\mathbf{Z}) - G(\mathbf{q},\mathbf{Z}) = \mathbf{q}'\mathbf{X}$	
(2)	$\partial F/\partial \mathbf{X} = \mathbf{q}$;	$\partial G/\partial \mathbf{q} = -\mathbf{X}$
(3)	$\mathbf{X} = -\partial G/\partial \mathbf{q}$;	$\mathbf{q} = \partial F/\partial \mathbf{X}$
(4)	$\partial F/\partial \mathbf{Z} = \partial G/\partial \mathbf{Z}$;	$\partial G/\partial \mathbf{Z} = \partial F/\partial \mathbf{Z}$
(5)	$F = G - \mathbf{q}'(\partial G/\partial \mathbf{q})$;	$G = F - \mathbf{X}'(\partial F/\partial \mathbf{X})$
(6)	$\mathbf{Z}$;	$\mathbf{Z}$

Under our assumptions on $F(\mathbf{X},\mathbf{Z})$, a Legendre transformation always exists. We introduce the Legendre transformation for a number of reasons. First, its use leads to a system of partial differential equations which may be used to either construct the normalized profit function explicitly or to study its behavior, given the production function and the first order necessary conditions for a maximum (and *vice versa*). Second, the Legendre transformation may be used to deduce equivalent structures of the production function and the normalized profit function. If the production function or the normalized profit function satisfies a given partial differential equation defining a certain structural property, then the same partial differential equation must also be satisfied by a Legendre transformation of variables. This is because we have shown that the production function and the normalized profit function are Legendre transformations of each other, hence a partial differential equation for $F(\mathbf{X},\mathbf{Z})$ in $\mathbf{X}$ and $\mathbf{Z}$ becomes a partial differential equation for $G(\mathbf{q},\mathbf{Z})$ in $\mathbf{q}$ and $\mathbf{Z}$. Thus, equivalent properties may be deduced immediately. This technique is used extensively in Sections 2 and 3. Third, the Legendre transformation may be useful in the solution of certain partial differential equations which may prove intractable otherwise. Suppose we wish to establish the class of production functions such that, under profit maximization, X_1/X_2 is constant. Starting from the set of functions

$$\frac{\partial F}{\partial X_1} = q_1 \quad \text{and} \quad \frac{\partial F}{\partial X_2} = q_2,$$

$$\frac{X_1}{X_2} = \frac{k_1}{k_2}.$$

This may appear to be rather intractable. However, by using the Legendre transformation, this problem becomes

$$\frac{\partial G/\partial q_1}{\partial G/\partial q_2} = \frac{k_1}{k_2},$$

with the general solution

$$G(\mathbf{q}) = g(k_1 q_1 + k_2 q_2),$$

which has a well-known dual

$$F(\mathbf{X}) = f\left(\min\left[\frac{X_1}{k_1}, \frac{X_2}{k_2}\right]\right).$$

Another example is furnished by the partial differential equation

$$X_1 = f_1\left(\frac{\partial F}{\partial X_1}, \frac{\partial F}{\partial X_2}\right).$$

By the Legendre transformation, this equation becomes

$$-\frac{\partial G}{\partial q_1} = f_1(q_1, q_2),$$

which may be integrated. This technique is used in Section 5.

We emphasize, however, that the Legendre transformations are procedures for studying expressions that are known to exist; they are not meant to be substitutes for the fundamental existence theorems for the dual functions, which are proved in Chapter I.1 for the general case, and in Sections 1.2 and 1.3 for the locally strongly concave case.

1.5. *Comparative Statics*

We present some comparative statics results that can be obtained directly by making use of the properties of normalized profit functions.

1.5.1. *Increase in nominal price of output*

(i) The optimal output is given by

$$Y^* = G(\mathbf{q},\mathbf{Z}) - \mathbf{q}'\frac{\partial G}{\partial \mathbf{q}}(\mathbf{q},\mathbf{Z}),$$

$$\frac{\partial Y^*}{\partial p} = \sum_k \frac{\partial G}{\partial q_k}\cdot\frac{\partial q_k}{\partial p} - \frac{\partial \mathbf{q}'}{\partial p}\frac{\partial G}{\partial \mathbf{q}}(\mathbf{q},\mathbf{Z}) - \mathbf{q}'\frac{\partial^2 G}{\partial \mathbf{q}\partial \mathbf{q}'}\frac{\partial \mathbf{q}}{\partial p}$$

$$= \frac{1}{p}\mathbf{q}'\frac{\partial^2 G}{\partial \mathbf{q}\partial \mathbf{q}'}\mathbf{q} > 0,$$

since

$$\left[\frac{\partial^2 G}{\partial \mathbf{q} \partial \mathbf{q}'}\right]$$

is positive definite. Thus the effect of output price on supply is positive.

(ii) $$\frac{\partial X_i^*}{\partial p} = \frac{\partial}{\partial p} - \frac{\partial G(\mathbf{q},\mathbf{Z})}{\partial q_i} = \frac{1}{p} \sum_{k=1}^{m} \frac{\partial^2 G}{\partial q_i \partial q_k} \cdot q_k,$$

which is not definite in sign.

1.5.2. *Increase in nominal price of a variable input*

(i) $$\frac{\partial Y^*}{\partial q_i^*} = \frac{1}{p}\left[\frac{\partial G}{\partial q_i}(\mathbf{q},\mathbf{Z}) - \sum_{k=1}^{m} \frac{\partial^2 G}{\partial q_i \partial q_k} q_k - \frac{\partial G}{\partial q_i}(\mathbf{q},\mathbf{Z})\right] = -\frac{1}{p}\left[\sum_{k=1}^{m} \frac{\partial^2 G}{\partial q_i \partial q_k} \cdot q_k\right],$$

which is again not definite in sign, but equal in magnitude but opposite in sign to $\partial X_i^* / \partial p$.

(ii) $$\frac{\partial X_i^*}{\partial q_i^*} = -\frac{1}{p} \frac{\partial^2 G}{\partial q_i^2} < 0,$$

since

$$\left[\frac{\partial^2 G}{\partial \mathbf{q} \partial \mathbf{q}'}\right]$$

is positive definite. Thus, the own price effect of input price on demand is negative.

(iii) $$\frac{\partial X_i^*}{\partial q_j^*} = -\frac{1}{p} \frac{\partial^2 G}{\partial q_i \partial q_j} = \frac{\partial X_j^*}{\partial q_i^*},$$

by the twice continuous differentiability of $G(\mathbf{q},\mathbf{Z})$. This is the well-known symmetry condition on cross-price effects.

(iv) By collecting these comparative statics results, we may derive, in addition, that

$$\sum_{k=1}^{m} \frac{\partial Y^*}{\partial q_i^*} \cdot q_i^* = -\frac{\partial Y^*}{\partial p} \cdot p < 0,$$

and

$$\sum_{k=1}^{m} \frac{\partial X_i^*}{\partial p} \cdot q_i^* = \frac{\partial Y^*}{\partial p} \cdot p > 0.$$

These results summarize the basic Hicksian Laws of Production.[11]

(v) It is important to note a relationship between the Hessian matrices of the production function and the normalized profit function. By differentiating

$$\frac{\partial F}{\partial \mathbf{X}} = \mathbf{q},$$

with respect to $\mathbf{q}$, treating $\mathbf{X}$ as implicit functions of $\mathbf{q}$, we have

$$\left[\frac{\partial \mathbf{X}}{\partial \mathbf{q}}\right]\left[\frac{\partial^2 F}{\partial \mathbf{X} \partial \mathbf{X}'}\right] = I,$$

but

$$\frac{\partial \mathbf{X}}{\partial \mathbf{q}} = -\left[\frac{\partial^2 G}{\partial \mathbf{q} \partial \mathbf{q}'}\right],$$

Thus

$$\frac{\partial^2 G}{\partial \mathbf{q} \partial \mathbf{q}'} = -\left[\frac{\partial^2 F}{\partial \mathbf{X} \partial \mathbf{X}'}\right]^{-1}.$$

Also, by differentiating $\partial F / \partial \mathbf{X} = \mathbf{q}$ with respect to $\mathbf{Z}$, we have

$$\left[\frac{\partial \mathbf{X}}{\partial \mathbf{Z}}\right]\left[\frac{\partial^2 F}{\partial \mathbf{X} \partial \mathbf{X}'}\right] + \left[\frac{\partial^2 F}{\partial \mathbf{Z} \partial \mathbf{X}'}\right] = 0,$$

or

$$\left[\frac{\partial^2 G}{\partial \mathbf{Z} \partial \mathbf{q}'}\right] = \left[\frac{\partial^2 F}{\partial \mathbf{Z} \partial \mathbf{X}'}\right]\left[\frac{\partial^2 F}{\partial \mathbf{X} \partial \mathbf{X}'}\right]^{-1}.$$

[11]See Hicks (1946, App.).

1.6. *Econometric Implementation*

Because of the derivative property of the normalized profit function, sometimes known as Hotelling's (1932) Lemma, namely,

$$\mathbf{X} = -\frac{\partial G}{\partial \mathbf{q}}, \qquad Y = G - \mathbf{q}'\frac{\partial G}{\partial \mathbf{q}},$$

the normalized profit function is especially useful for the purpose of econometric specification of supply and demand functions. With the normalized profit function, it is not necessary to actually solve a profit maximization problem. As long as one starts out with a normalized profit function which satisfies Assumptions (G.1) through (G.7), one is assured that the supply and demand functions obtained through differentiation of G are consistent with profit maximization subject to a production function and given normalized prices. In particular, since one is free to choose the functional form of $G(q,Z)$, one may choose a parametric form that is most convenient from the point of view of econometric estimation.

There are two other points worth mentioning. First, as McFadden has stressed, convexity of the profit function is a consequence of profit maximization and does not depend at all on the concavity of the production function, so long as a proper profit function exists and is attainable for at least one set of prices. Hence, if one is willing to maintain the assumption of profit maximization, it is not necessary to insist that the production function is concave. Second, for the purpose of estimating the normalized profit function parameters, one should use all of the stochastically independent supply and demand functions for maximum efficiency. This in general entails, because of symmetry of cross-price effects, restrictions across equations.

Finally, one should also add at this point that for many empirical applications in which the observed range of normalized prices is a compact and convex set, it may not be necessary to require that the normalized profit function should satisfy Assumptions (G.1) through (G.7) globally, that is, for all possible prices. It is in many instances sufficient to have the Assumptions (G.1) through (G.7) hold locally within a compact and convex set. As long as interest is focused on this convex set, a normalized profit function, although not globally valid, may nevertheless provide an adequate local approximation. In particular, one can often modify such a function so that it satisfies globally the weak regularity conditions for normalized profit functions given in Chapter I.1.

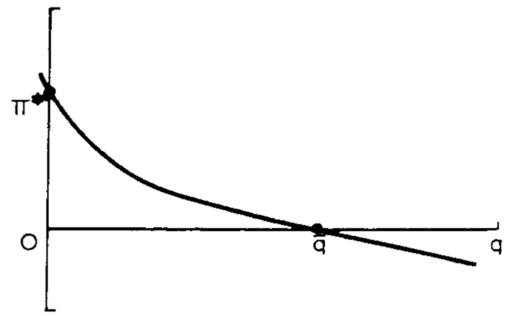

FIGURE 1

We shall illustrate the modification technique with an example. Suppose that the normalized profit function has the form shown in Figure 1 – sloping downward all the way. This function is decreasing and convex in q, but it is not non-negative as a normalized profit function should be. It is also defined for negative prices. One may modify this function so that

$$\Pi^* \text{ is not defined for } \quad q < 0,$$
$$\Pi^* = 0 \quad \text{for} \quad q \geqq \bar{q}.$$

With this modification, the normalized profit function satisfies the usual regularity conditions (such as, for instance, those given in Chapter I.1). As long as the domain of interest is contained in the open interval $(0, \bar{q})$ this normalized profit function will serve just as well as other normalized profit functions which satisfy the regularity conditions globally without modifications of the type considered here.

2. The Structure of Normalized Profit Functions

2.1. *The Case of a Single Output*

For purposes of applications, it is useful to know what are equivalent properties for the production function and the corresponding normalized profit function. To this end, we state and prove several theorems relating equivalent structures of production functions and normalized profit functions.

Theorem II-1. Under Assumptions (F.1) through (F.7), a production function is homogeneous of degree k in $\mathbf{X}$ if and only if the normalixed profit function is homogeneous of degree $-(k/(1-k))$ in $\mathbf{q}$.

Definition. A function is *homogeneous of degree k* in **X** if

$$F(\lambda \mathbf{X},\mathbf{Z}) = \lambda^k F(\mathbf{X},\mathbf{Z}), \quad \text{for} \quad \forall\lambda > 0, \quad \forall \mathbf{Z} \in \bar{R}^n_+, \quad \forall \mathbf{X} \in \bar{R}^m_+.$$

Proof: This follows directly from the dual transformation properties. By Euler's Theorem for homogeneous functions,

$$\sum_{i=1}^{m} \frac{\partial F}{\partial X_i} X_i = kF. \tag{II-1}$$

Applying the dual transformation, equation (II-1) becomes

$$-\sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i} = k\left(G - \sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i}\right).$$

Therefore,

$$\sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i} = -\frac{k}{(1-k)} G.$$

Hence, by Euler's Theorem, G is homogeneous of degree $-k/(1-k)$ in **q**. The converse is proved similarly. Observe that the case $k \geqq 1$ violates the local strong concavity assumption. Q.E.D.

Corollary 1.1. Under Assumptions (F.1) through (F.7), and homogeneity of degree k of $F(\mathbf{X},\mathbf{Z})$ in **X**,

$$Y^* = (1-k)^{-1} G,$$

and

$$C^* = p \frac{k}{(1-k)} G,$$

where C^* is the profit-maximizing cost of the variable inputs.

Proof: By the dual transformation

$$\begin{aligned} Y^* &= \left(G - \sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i}\right) \\ &= G + \frac{k}{(1-k)} G \\ &= (1-k)^{-1} G \end{aligned}$$

$$C^* = P \sum_{i=1}^{m} q_i \left(-\frac{\partial G}{\partial q_i}\right) = p \frac{k}{(1-k)} G. \quad \text{Q.E.D.}$$

This corollary implies that for a homogeneous production function the profit-maximizing output is proportional to normalized profit. In other words, profit-maximizing revenue is proportional to profit-maximizing profit. Likewise profit-maximizing cost is also proportional to profit-maximizing profit. These are clearly testable consequences of the homogeneity assumption.

Corollary 1.2. Under Assumptions (F.1) through (F.7), the derived demand functions are homogeneous of degree $-1/(1-k)$ in $\mathbf{q}$ if the production function is homogeneous of degree k in $\mathbf{X}$.

Proof: This follows directly from the fact that the demand functions are derivatives of the normalized profit function, which is homogeneous of degree $-(k/(1-k))$. Q.E.D.

The concept of homogeneity has been generalized by Shephard (1953 and 1970) to that of homotheticity. We give a definition that is closely related but slightly different from his.

Definition. A function is *homothetic* in $\mathbf{X}$ if it can be written in the form

$$F(H(\mathbf{X},\mathbf{Z}),\mathbf{Z}),$$

where for each $\mathbf{Z} \in \bar{R}^n_+$, F is a positive, finite, continuous and strictly monotonic function of one variable H with $F(0,\mathbf{Z}) = 0$, and H is a homogeneous function of degree one in $\mathbf{X}$.

An important property of homothetic functions is the following:

Lemma II-1. A function with strictly non-zero first partial derivatives is homothetic in $\mathbf{X}$ if and only if the ratio of each possible pair of first partial derivatives with respect to $\mathbf{X}$ is a homogeneous function of degree zero in $\mathbf{X}$.

This lemma is proved in Lau (1969a) and will not be repeated here. Based on Lemma II-1, we state and prove the following theorem:

Theorem II-2. Under Assumptions (F.1) through (F.7), a production function is homothetic in $\mathbf{X}$ if and only if the normalized profit function is homothetic in $\mathbf{q}$.

Proof: For a homothetic production function the first-order necessary conditions for a maximum imply that

$$\frac{\partial F/\partial H}{\partial F/\partial H}\frac{\partial H/\partial X_i}{\partial H/\partial X_1}=\frac{q_i}{q_1}, \quad \forall i.$$

By homotheticity, the left-hand side of the equation is homogeneous of degree zero in $\mathbf{X}$. One may therefore rewrite the left-hand side as functions only of X_i/X_1. Our Assumptions (F.1) through (F.7) are sufficient to ensure that the (X_i/X_1)'s may be solved uniquely as continuously differentiable functions of (q_i/q_1)'s,

$$\frac{X_i}{X_1}=f_i^1\left(\frac{q_2}{q_1},\frac{q_3}{q_1},\ldots,\frac{q_m}{q_1},\mathbf{Z}\right), \quad \forall i,$$

which by using the dual transformation yields

$$\frac{\partial G/\partial q_i}{\partial G/\partial q_1}=f_i^1\left(\frac{q_2}{q_1},\frac{q_3}{q_1},\ldots,\frac{q_m}{q_1},\mathbf{Z}\right).$$

Since the ratios of the first partial derivatives of G with respect to $\mathbf{q}$ are homogeneous of degree zero in $\mathbf{q}$, G is homothetic in $\mathbf{q}$ by Lemma II-1. The converse is proved similarly starting from

$$\frac{\partial G/\partial H}{\partial G/\partial H}\frac{\partial H/\partial q_i}{\partial H/\partial q_1}=\frac{X_i}{X_1}. \qquad \text{Q.E.D.}$$

The next theorem shows the effect of changing the scale of measurement of output (or, as some authors prefer it, the level of technical efficiency):

Theorem II-3. Let $Y=F(\mathbf{X},\mathbf{Z})$ and $\Pi^*=G(\mathbf{q},\mathbf{Z})$ be a production function satisfying Assumptions (F.1) through (F.7) and its conjugate normalized profit function, respectively. Then for any $A>0$, if the production function is given by $Y=AF(\mathbf{X},\mathbf{Z})$, the normalized profit function is given by $\Pi^*=AG(\mathbf{q}/A,\mathbf{Z})$.[12]

Proof:

$$\begin{aligned}\Pi^* &= \max\{AF(\mathbf{X},\mathbf{Z})-\mathbf{q}'\mathbf{X}\}\\ &= A\max\{F(\mathbf{X},\mathbf{Z})-\mathbf{q}'/A\mathbf{X}\}\\ &= AG(\mathbf{q}/A,\mathbf{Z}). \qquad \text{Q.E.D.}\end{aligned}$$

[12]This theorem is proved in Fenchel (1953, pp. 93–94); see also Chapter I.1, Table 2, composition rule 1. This theorem is a special case of Theorem 28 in Chapter I.1.

The next theorem shows the effect of a translation of the origin:

Theorem II-4. Let $Y = F(\mathbf{X},\mathbf{Z})$ and $\Pi^* = G(\mathbf{q},\mathbf{Z})$ be a production function satisfying Assumptions (F.1) through (F.7) and its conjugate normalized profit function, respectively. Then for any constant $\bar{Y} > 0$ and constant vector $\bar{\mathbf{X}} > 0$, if the production function is given by $Y = \bar{Y} + F(\mathbf{X}+\bar{\mathbf{X}},\mathbf{Z})$, the normalized profit function is given by $\Pi^* = \bar{Y} + G(\mathbf{q},\mathbf{Z}) + \mathbf{q}'\bar{\mathbf{X}}$.[13]

Proof:

$$\begin{aligned}\Pi^* &= \max_{\mathbf{X}}\{\bar{Y} + F(\mathbf{X}+\bar{\mathbf{X}},\mathbf{Z}) - \mathbf{q}'\mathbf{X}\}\\ &= \bar{Y} + \max_{\mathbf{X}^*}\{F(\mathbf{X}^*,\mathbf{Z}) - \mathbf{q}'(\mathbf{X}^* - \bar{\mathbf{X}})\}\\ &= \bar{Y} + \mathbf{q}'\bar{\mathbf{X}} + \max_{\mathbf{X}^*}\{F(\mathbf{X}^*,\mathbf{Z}) - \mathbf{q}'\mathbf{X}^*\}\\ &= \bar{Y} + G(\mathbf{q},\mathbf{Z}) + \mathbf{q}'\bar{\mathbf{X}}. \qquad \text{Q.E.D.}\end{aligned}$$

Theorem II-5. Under Assumptions (F.1) through (F.7), let the sum of production elasticities be given as

$$\epsilon = \sum_{i=1}^{m} \frac{\partial \ln F}{\partial \ln X_i},$$

then

$$\frac{\partial \ln G}{\partial \ln q_i} = -\frac{1}{(1-\epsilon)}\frac{\partial \ln F}{\partial \ln X_i}.$$

Proof: By the dual transformation,

$$\frac{G}{F} = (1-\epsilon).$$

Strong concavity implies that $\epsilon < 1$. Thus,

$$\begin{aligned}\frac{\partial \ln G}{\partial \ln q_i} &= \frac{q_i}{G}\frac{\partial G}{\partial q_i}\\ &= \frac{\partial F/\partial X_i}{(1-\epsilon)F}(-X_i),\end{aligned}$$

[13] This theorem is proved in Fenchel (1953, pp. 94–95).

by the dual transformation. Thus,

$$\frac{\partial \ln G}{\partial \ln q_i} = -\frac{1}{(1-\epsilon)}\frac{\partial \ln F}{\partial \ln X_i}. \qquad \text{Q.E.D.}$$

Corollary 5.1. Under Assumptions (F.1) through (F.7), let the sum of normalized profit elasticities be given as

$$\eta = \sum_{i=1}^{m} \frac{\partial \ln G}{\partial \ln q_i},$$

then

$$\frac{\partial \ln F}{\partial \ln X_i} = -\frac{1}{(1-\eta)}\frac{\partial \ln G}{\partial \ln q_i}. \qquad \text{Q.E.D.}$$

Proof: Identical to the theorem.

Corollary 5.2. Under Assumptions (F.1) through (F.7),

$$\frac{1}{\epsilon} + \frac{1}{\eta} = 1.$$

Proof: By the theorem,

$$\begin{aligned} \sum_{i=1}^{m} \frac{\partial \ln G}{\partial \ln q_i} &= -\frac{1}{(1-\epsilon)} \sum_{i=1}^{m} \frac{\partial \ln F}{\partial \ln X_i} \\ &= -\frac{1}{(1-\epsilon)}\epsilon \\ &= \eta. \end{aligned}$$

Thus,

$$-\epsilon = \eta - \eta\epsilon, \qquad \eta + \epsilon = \eta\epsilon.$$

Dividing through by $\eta\epsilon$, we obtain

$$\frac{1}{\epsilon} + \frac{1}{\eta} = 1. \qquad \text{Q.E.D.}$$

Theorem II-5 shows how estimates of production elasticities may be derived given the estimates of the normalized profit elasticities and vice versa.

Theorem II-6. Under Assumptions (F.1) through (F.7), a homogeneous production function of degree k in $\mathbf{X}$, $0<k<1$, is separable with respect to a commodity-wise partition in $\mathbf{X}$ if and only if the normalized profit function is also separable price-wise in $\mathbf{q}$.

Proof: Homogeneity implies that $F(\mathbf{X},\mathbf{Z}) = H(\mathbf{X},\mathbf{Z})$, where H is a homogeneous function of degree k in $\mathbf{X}$. Separability implies that

$$\frac{\partial}{\partial X_k}\left(\frac{(\partial H/\partial X_i)(\mathbf{X},\mathbf{Z})}{(\partial H/\partial X_j)(\mathbf{X},\mathbf{Z})}\right) = 0, \qquad \forall i,j,k, \quad i \neq j \neq k,$$

which in turn implies that $(\partial H/\partial X_i)/(\partial H/\partial X_j)$ is a function of only X_i, X_j and $\mathbf{Z}$. Homogeneity implies that this function is homogeneous of degree zero. Thus, one has

$$\begin{aligned}\frac{q_i}{q_j} = \frac{\partial H/\partial X_i}{\partial H/\partial X_j} &= h_{ij}(X_i,X_j,\mathbf{Z}) \\ &= h_{ij}\left(\frac{X_i}{X_j},1,\mathbf{Z}\right).\end{aligned}$$

If this equation can be solved for X_i/X_j as a function of q_i/q_j and $\mathbf{Z}$, then it follows immediately by a dual transformation that $(\partial G/\partial q_i)/(\partial G/\partial q_j)$ is independent of $q_k, k \neq i,j$. But Assumptions (F.1) through (F.7) are sufficient to guarantee that X_i/X_j are continuously differentiable functions of $\mathbf{q}$. Thus, the function (X_i/X_j) exists and we conclude that G is separable price-wise.

The converse of this theorem may be proved in a similar manner by observing that G is also homogeneous by Theorem II-1. This completes the proof. Q.E.D.

Corollary 6.1. Under Assumptions (F.1) through (F.7), a production function homothetic in $\mathbf{X}$ is separable commodity-wise in $\mathbf{X}$ if and only if the normalized profit function is separable price-wise in $\mathbf{q}$.

Proof: This follows directly from Theorems II-3 and II-6, and Lemma II-1. Q.E.D.

We need the following lemma to prove a generalized version of Theorem II-6 which applied to production functions in which the inputs may be grouped into several categories, such as capital and labor, each of which may consist of capital and labor of many different kinds.

Lemma II-2. A strongly separable function is homothetic if and only if each category function (or quantity index) is homogeneous of the same degree, or it is a function of products of homogeneous category functions.

This lemma implies that if Y is strongly separable, that is, if

$$Y = F\left(\sum_{i=1}^{m} X^i(X_{i1},\dots,X_{in},\mathbf{Z})\right)$$

then Y is homothetic if and only if either each X^i is homogeneous of the same degree or $Y = F(\prod_{i=1}^{m} X^i(X_{i1},\dots,X_{in},\mathbf{Z}))$, where each X^i is homogeneous (not necessarily of the same degree). This is proved in Lau (1969a). We omit the proof.

Theorem II-7. Under Assumptions (F.1) through (F.7), a production function is additively separable with respect to the commodity categories if and only if the normalized profit function is additively separable with respect to the corresponding price categories.

Proof: Additive separability implies

$$Y = \sum_{i=1}^{m} X^i(X_{i1},\dots,X_{in_i},\mathbf{Z}).$$

It is easy to see the maximization of

$$P^* = Y - \sum_{i=1}^{m}\sum_{j=1}^{n_j} q_{ij}X_{ij}$$

results in demand functions for X_{ij}'s which depend only on the normalized prices of the commodities of the ith category and $\mathbf{Z}$. Thus $G(\mathbf{q},\mathbf{Z})$ must also be additively separable in $\mathbf{q}$. The converse is proved similarly. Q.E.D.

Theorem II-8. Under Assumptions (F.1) through (F.7), a production function is homogeneous and strongly separable with respect to the commodity categories if and only if the normalized profit function is homogeneous and strongly separable with respect to the corresponding price categories.

Proof: Homogeneity follows from Theorem II-1. The first-order necessary conditions for a maximum require that

$$\frac{\partial F/\partial X_{ir}}{\partial F/\partial X_{js}} = \frac{\partial X^i/\partial X_{ir}}{\partial X^j/\partial X_{js}}$$
$$= \frac{X^i_r(X_{i1},\ldots,X_{in_i},\mathbf{Z})}{X^j_s(X_{j1},\ldots,X_{jn_j},\mathbf{Z})} = \frac{q_{ir}}{q_{js}}, \qquad i \neq j, \quad i,j = 1,\ldots,m,$$
$$= \frac{X^i_r(X_{i1}/X_{j1},\ldots,X_{in_i}/X_{j1},\mathbf{Z})}{X^j_s(1,\ldots,X_{jn_j}/X_{j1},\mathbf{Z})}, \qquad r = 1,\ldots,n_i, \quad s = 1,\ldots,n_j,$$

by zero degree homogeneity. Note that there exists $n_i + n_j - 1$ independent equations for each pair (i,j) in the $n_i + n_j - 1$ unknown X_{ir}/X_{j1}'s and X_{js}/X_{j1}'s. Moreover, from Lemma I-2, the optimal factor proportions are continuously differentiable function of only q_{ir}/q_{j1}'s and q_{js}/q_{j1}'s. Hence one has

$$\frac{\partial}{\partial q_{kt}}(X_{ir}/X_{j1}) = 0, \qquad k \neq i,j,$$

which implies also that

$$\frac{\partial}{\partial q_{kt}}(X_{ir}/X_{js}) = 0, \qquad k \neq i,j. \tag{II-2}$$

On applying the dual transformation, equation (II-2) becomes

$$\frac{\partial}{\partial q_{kt}}\left(\frac{\partial G/\partial q_{ir}}{\partial G/\partial q_{js}}\right) = 0, \qquad k \neq i,j, \quad i,j,k = 1,\ldots,m, \quad r = 1,\ldots,n_i, \quad s = 1,\ldots,n_j, \quad t = 1,\ldots,n_k.$$

Hence G is strongly separable. The converse is proved similarly. Q.E.D.

Corollary 8.1. Under Assumptions (F.1) through (F.7), a production function is homothetic and strongly separable if and only if the normalized profit function is homothetic and strongly separable.

Proof: Homotheticity follows from Theorem II-2. Otherwise essentially the same proof of the theorem suffices. Q.E.D.

Note the crucial role of the homogeneity of each category function. Otherwise it will not be possible to express X_{ir}/X_{js} as a function of only $\{q_{it},q_{jt}\}$.

Definition. A function is said to be *homothetically separable* if it is weakly separable and each category function is homothetic. (Note that the function itself need not be homothetic.)

We now introduce Lemma II-3, which is also proved in Lau (1969a).

Lemma II-3. A homothetic and weakly separable function is homothetically separable.

Theorem II-9. Under Assumptions (F.1) through (F.7), a production function is homothetically separable if and only if the normalized profit function is homothetically separable.

Proof: The first order necessary conditions are

$$\frac{\partial X^i/\partial X_{ir}}{\partial X^i/\partial X_{i1}} = \frac{X^i_r(1,\ldots,X_{in_i}/X_{i1},\mathbf{Z})}{X^i_1(1,\ldots,X_{in_i}/X_{i1},\mathbf{Z})} = \frac{q_{ir}}{q_{i1}}, \qquad r = 2,\ldots,n_i.$$

Thus by an argument similar to that in previous theorems one has

$$\frac{\partial}{\partial q_{jt}}\left(\frac{\partial G/\partial q_{ir}}{\partial G/\partial q_{is}}\right) = 0, \qquad j \neq i, \quad \forall r,s,t.$$

The converse is proved similarly. Q.E.D.

Corollary 9.1. Under Assumptions (F.1) through (F.7), a production function is homothetic and weakly separable if and only if the normalized profit function is homothetic and weakly separable.

Proof: This follows from Lemma II-3 and the Theorem II-9. Q.E.D.

Theorem II-10. Under Assumptions (F.1) through (F.7), a production function and its normalized profit function are strongly separable (but not additively separable) with respect to the commodity categories and the corresponding price categories respectively only if they are homothetic.

Proof: Strong separability of both F and G implies

$$Y = F\left(\sum_{i=1}^{m} X^i(X_{i1},\ldots,X_{in_i},\mathbf{Z})\right)$$

and

$$\Pi^* = G\left(\sum_{i=1}^{m} Q^i(q_{i1},\ldots,q_{in_i},\mathbf{Z})\right).$$

Now

$$\frac{\partial F/\partial X_{ir}}{\partial F/\partial X_{js}} - \frac{X^i_r(X_{i1},\ldots,X_{in_i},\mathbf{Z})}{X^j_s(X_{j1},\ldots,X_{jn_j},\mathbf{Z})}. \quad \text{(II-3)}$$

Applying the dual transformation to equation (II-3), we have

$$\frac{q_{ir}}{q_{js}} = \frac{X^i_r(-\partial G/\partial q_{i1},\ldots,-\partial G/\partial q_{in_i},\mathbf{Z})}{X^j_s(-\partial G/\partial q_{j1},\ldots,-\partial G/\partial q_{jn_j},\mathbf{Z})}. \quad \text{(II-4)}$$

Differentiating both sides of equation (II-4) by q_{kt}, $k \neq i,j$, and observing that

$$\frac{\partial^2 G}{\partial q_{ir}\partial q_{kt}} = G'' Q^i_r Q^k_t,$$

equation (II-4) becomes

$$X^j_s \sum_{l=1}^{n_i} X^i_{rl} \cdot G'' Q^i_l Q^k_t - X^i_r \sum_{l=1}^{n_j} X^j_{sl} \cdot G'' Q^j_l Q^k_t = 0. \quad \text{(II-5)}$$

Now $G'' \neq 0$, otherwise the production function is additive in the X^i's by Theorem II-7, which is ruled out by hypothesis. Moreover, observe that

$$G' Q^i_l = -X_{il}.$$

Hence, equation (II-5) becomes, after multiplication by $G'/G'' Q^k_t$,

$$\sum_{l=1}^{n_i} \frac{\partial}{\partial X_{il}} \left(\frac{X^i_r}{X^j_s}\right) \cdot X_{il} + \sum_{l=1}^{n_j} \frac{\partial}{\partial X_{jl}} \left(\frac{X^i_r}{X^j_s}\right) \cdot X_{jl} = 0.$$

By Euler's Theorem, (X^i_r/X^j_s) is homogeneous of degree zero in $\mathbf{X}$, $\forall i,j,r,s$. By Lemma II-1, F is homothetic, and by Theorem II-1, G is also homothetic. Q.E.D.

Note that by Lemma II-2 then, the X_i's are either homogeneous of the same degree or are logarithms of homogeneous functions.

Theorem II-11. Under Assumptions (F.1) through (F.7), a production function and its corresponding normalized profit function are both weakly separable only if they are both homothetically separable.

Proof: Let the production and normalized profit functions be

$$Y = F(X^1(X_{11},\ldots,X_{1n_1},\mathbf{Z}),\ldots,X^m(X_{m1},\ldots,X_{mn_m},\mathbf{Z}),\mathbf{Z}),$$

and

$$\Pi^* = G(Q^1(q_{11},\ldots,q_{1n_1},\mathbf{Z}),\ldots,Q^m(q_{m1},\ldots,q_{mn_m},\mathbf{Z}),\mathbf{Z}).$$

It is necessary to show that each X^i and hence each G^i is homothetic. The proof is strictly analogous to that of Theorem II-10. Applying the dual transformation to the first-order necessary condition, one obtains

$$\frac{q_{ir}}{q_{is}} = \frac{X^i_r(-\partial G/\partial q_{i1},\ldots,-\partial G/\partial q_{in_i},\mathbf{Z})}{X^i_s(-\partial G/\partial q_{i1},\ldots,-\partial G/\partial q_{in_i},\mathbf{Z})}. \tag{II-6}$$

Differentiating both sides of equation (II-6) by q_{jt}, $j \neq i$, we have

$$X^i_s \sum_{l=1}^{n_i} X^i_{rl}\left(-\frac{\partial^2 G}{\partial q_{il}\partial_{jt}}\right) - X^i_r \sum_{l=1}^{n_i} X^i_{sl}\left(-\frac{\partial^2 G}{\partial q_{il}\partial q_{jt}}\right) = 0. \tag{II-7}$$

For a weakly separable normalized profit function

$$\frac{\partial^2 G}{\partial q_{il}\partial q_{jt}} = Q^i_l \frac{\partial^2 G}{\partial Q^i \partial Q^j} Q^j_t = Q^i_l \cdot G_{ij} \cdot Q^j_t.$$

Hence, equation (II-7) becomes

$$\sum_{l=1}^{n_i} (X^i_s X^i_{rl} - X^i_r X^i_{sl}) \cdot X_{il} \cdot \left(\frac{G_{ij}}{G_i}\right) Q^j_t = 0.$$

Now, $G_i \neq 0$ and $G_{ij} \neq 0$, the latter because of weak (but not strong) separability. Therefore,

$$\sum_{l=1}^{n_i} \frac{\partial}{\partial X_{il}}\left(\frac{X^i_r}{X^i_s}\right) \cdot X_{il} = 0.$$

By Euler's Theorem, (X^i_r/X^i_s) is homogeneous of degree zero, $\forall r,s$. By Lemma II-1 each X^i is homothetic. By Theorem II-9, each Q^i is homothetic. Q.E.D.

These theorems are useful in specifying technologies with multiple variable input categories. They also have application in aggregation, in the construction of quantity and price indices and in the analysis of organization and information structures.

It should be noted that homogeneity, separability and other similar properties of the normalized profit function considered here may be alternatively deduced through the cost function by utilizing the general composition rules for cost functions in Theorem 9 in Chapter I.1 along with direct arguments on maximization of $Y - C(Y,q,Z)$ where C is the cost function. Here we have relied primarily on the Legendre transformation because the proofs are more direct and immediate. Of course, the proofs only apply under conditions which allow the use of the Legendre transformation, for example, under Assumptions (F.1) through (F.7) on the production function.

2.2. *Structures Involving Fixed Inputs*

Thus far we have not examined structural properties which involve the fixed inputs **Z**. Normalized profit functions with fixed inputs are sometimes referred to as normalized restricted profit functions [see Lau (1976a)]. To analyze structures involving fixed inputs, we introduce the concept of almost homogeneity.

Definition. A function $F(X,Z)$ is *almost homogeneous of degrees* k_1 *and* k_2 in **X** and **Z**, respectively, if

$$F(\lambda \mathbf{X},\lambda^{k_2}\mathbf{Z}) = \lambda^{k_1}F(\mathbf{X},\mathbf{Z}), \qquad \forall\lambda > 0.^{14} \tag{II-8}$$

The economic interpretation of an almost homogeneous production function is the following: if a set of inputs **X** is increased by the same proportion and another set of inputs **Z** is increased by some power of that proportion, then output Y will be increased by another power of that proportion. In the special case that $k_1 = k_2 = 1$, we have constant returns to scale in all inputs.

It will be shown that an almost homogeneous function satisfies a modified Euler's Theorem.

Lemma II-4. A continuously differentiable function is almost homogeneous of degree k_1 and k_2 if and only if

$$\sum_{i=1}^{m} \frac{\partial F}{\partial X_i}\cdot X_i + k_2 \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}\cdot Z_i = k_1 F(\mathbf{X},\mathbf{Z}). \tag{II-9}$$

Proof:
Necessity. If $F(\mathbf{X},\mathbf{Z})$ is almost homogeneous it satisfies equation (II-8). Differentiation of equation (II-8) with respect to λ yields

$$\sum_{i=1}^{m} \frac{\partial F}{\partial X_i}\cdot X_i + k_2 \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}\, \lambda^{k_2-1}\cdot Z_i = k_1\lambda^{k_1-1}F(\mathbf{X},\mathbf{Z}).$$

[14] See Aczel (1966, Ch. 7) for a discussion of almost homogeneous functions. Lau (1972) defines almost homogeneity in a slightly different manner.

This must hold identically for all $\lambda > 0$ and in particular for $\lambda = 1$. Hence

$$\sum_{i=1}^{m} \frac{\partial F}{\partial X_i} \cdot X_i + k_2 \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i} \cdot Z_i = k_1 F(\mathbf{X},\mathbf{Z}).$$

Sufficiency. Suppose $F(\mathbf{X},\mathbf{Z})$ satisfies equation (II-9). We note that

$$\frac{\partial F}{\partial Z_i} = \frac{\partial F}{\partial Z_i^{1/k_2}} \cdot \frac{1}{k_2} Z_i^{1/k_2 - 1},$$

one may therefore rewrite equation (II-9) in the form

$$\sum_{i=1}^{m} \frac{\partial F}{\partial X_i} X_i + \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i^{1/k_2}} \cdot Z_i^{1/k_2} = k_1 F(\mathbf{X},\mathbf{Z}).$$

Thus, by Euler's Theorem, F must be homogeneous of degree k_1 in $\mathbf{X}$ and $\mathbf{Z}^{1/k_2}$. In other words,

$$F(\mathbf{X},\mathbf{Z}) = H(\mathbf{X},\mathbf{Z}^{1/k_2}),$$

where H is homogeneous of degree k_1.

$$\begin{aligned} F(\lambda\mathbf{X},\lambda^{k_2}\mathbf{Z}) &= H(\lambda\mathbf{X},(\lambda^{k_2}\mathbf{Z})^{1/k_2}) \\ &= H(\lambda\mathbf{X},\lambda\mathbf{Z}^{1/k_2}) \\ &= \lambda^{k_1} F(\mathbf{X},\mathbf{Z}). \qquad \text{Q.E.D.} \end{aligned}$$

Theorem II-12. Under Assumptions (F.1) through (F.7), a production function is homogeneous of degree k in all inputs, variable and fixed, if and only if the normalized restricted profit function is almost homogeneous of degrees $-1/(1-k)$ and $-k/(1-k)$ if $k \neq 1$, and homogeneous of degree one in $\mathbf{Z}$ if $k = 1$.

Proof: By Euler's Theorem,

$$\sum_{i=1}^{m} \frac{\partial F}{\partial X_i} \cdot X_i + \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i} \cdot Z_i = kF.$$

By a dual transformation, one has

$$-\sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i} + \sum_{i=1}^{n} \frac{\partial G}{\partial Z_i} \cdot Z_i = k\left(G - \sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i}\right),$$

which simplifies to

$$\sum_{i=1}^{m} \frac{\partial G}{\partial q_i} q_i - \frac{1}{(1-k)} \sum_{i=1}^{n} \frac{\partial G}{\partial Z_i} Z_i = -\frac{k}{(1-k)} G, \quad \text{if} \quad k \neq 1,$$

or to

$$\sum_{i=1}^{n} \frac{\partial G}{\partial Z_i} \cdot Z_i = G, \qquad \text{if} \quad k = 1.$$

The converse is proved by retracing the steps. Q.E.D.

Note that $k > 1$ implies increasing returns to scale in all inputs. For the purpose of this theorem k may be either greater than or less than one.

Corollary 12.1. Under Assumptions (F.1) through (F.7), a production function is homogeneous of degree k in **Z**, $k > 0$, if and only if the normalized restricted profit function is almost homogeneous of degrees 1 and $1/k$.

Proof: By Euler's Theorem,

$$\sum_{i=1}^{n} \frac{\partial F}{\partial Z_i} \cdot Z_i = kF.$$

By a dual transformation, one has

$$\sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i} + \frac{1}{k} \sum_{i=1}^{n} \frac{\partial G}{\partial Z_i} Z_i = G.$$

Thus, G is almost homogeneous of degrees 1 and $1/k$. The converse is proved similarly. Q.E.D.

Corollary 12.2. Under Assumptions (F.1) through (F.2), and homogeneity of degree k, $k \neq 1$ in all inputs, the derived demand functions are almost homogeneous of degrees $-1/(1-k)$ and $-k/(1-k)$.

Proof: This follows from differentiating the next to the last equation in the proof of the theorem. Q.E.D.

Next we wish to characterize the normalized restricted profit function corresponding to a homothetic production function. A production function is homothetic in **X** and **Z** if it can be written in the form

$$Y = F(H(\mathbf{X},\mathbf{Z})),$$

where F is a positive, finite, continuous and strictly monotonic function of one variable with $F(0) = 0$ and H is a homogeneous function of degree one in **X** and **Z**. Homogeneity of H implies that $H(\mathbf{0},\mathbf{0}) = 0$. If $F(\cdot,\cdot)$ is non-negative and strictly *monotonically* increasing on $R_+^m \times \bar{R}_+^n$, then one can always choose F and H such that $F(\cdot)$ and $H(\cdot,\cdot)$ are both non-negative and strictly increasing on the non-negative real line and $R_+^m \times \bar{R}_+^n$, respectively. Monotonicity of $F(.,.)$ implies that $F(\cdot)$ and $H(.,.)$ must be monotonic in the same direction. Subject to the convention that $H(\cdot,\cdot)$ is strictly increasing on $R_+^m \times \bar{R}_+^n$, Euler's Theorem requires that $H(\cdot,\cdot)$ be non-negative on $R_+^m \times \bar{R}_+^n$. Thus both $F(\cdot)$ and $H(\cdot,\cdot)$ can be chosen to be non-negative and strictly increasing on $\bar{R}_+$ and $R_+^m \times \bar{R}_+^n$, respectively. Given $F(0) = 0$, this implies that $F(\cdot)$ will be non-negative on $\bar{R}_+$.

We introduce Lemma II-5:

Lemma II-5. Under Assumptions (F.1) through (F.7), a production function is homothetic in **X** and **Z** if and only if

$$\sum_{i=1}^{m} \frac{\partial F}{\partial X_i} \cdot X_i + \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i} \cdot Z_i = f(F(X,Z)),$$

where f is an arbitrary, finite, non-negative function of a single variable with $f(0) = 0$, continuous on $\bar{R}_+$ and continuously differentiable on R_+.

A proof of a similar result is available in Lau (1969a). We omit the proof.

Theorem II-13. Under Assumptions (F.1) through (F.7), a production function is homothetic in **X** and **Z** if and only if the normalized profit function satisfies the equation

$$-\sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i} + \sum_{i=1}^{n} \frac{\partial G}{\partial Z_i} \cdot Z_i = f\left(G - \sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i}\right), \tag{II-10}$$

where f is an arbitrary, finite, non-negative function of a single variable with $f(0) = 0$, continuous on $\bar{R}_+$ and continuously differentiable on R_+.

Proof: The proof is immediate using Lemma II-5 and the dual transformation

$$\frac{\partial F}{\partial X_i} = q_i, \qquad \frac{\partial F}{\partial Z_i} = \frac{\partial G}{\partial Z_i},$$

$$X_i = -\frac{\partial G}{\partial q_i}, \qquad F(\mathbf{X},\mathbf{Z}) = G - \sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i}. \qquad \text{Q.E.D.}$$

There remains the question of a nomenclature for the class of functions defined by equation (II-10). We know from Theorem II-12 that homogeneous functions are dual to almost homogeneous functions. We shall refer to functions satisfying equation (II-10) as *partially homothetic* functions.

Theorem II-14. Under Assumptions (F.1) through (F.7), a production function is homothetically separable in **X**, that is,

$$Y = F(H(\mathbf{X}),\mathbf{Z}),$$

where H is a homogeneous function of degree one if and only if the normalized restricted profit function is homothetically separable, that is,

$$\Pi^* = G(H^*(\mathbf{q}),\mathbf{Z}),$$

where H^* is a homogeneous function of degree one.

Proof: This follows from Theorem II-2. Q.E.D.

Corollary 14.1. Under Assumptions (F.1) through (F.7), a production function has the form

$$Y = F(H(\mathbf{X}),f(\mathbf{Z}))$$

where H is a homogeneous function of degree one if and only if the normalized restricted profit function has the form

$$\Pi^* = G(H^*(\mathbf{q}),f(\mathbf{Z})),$$

where H^* is a homogeneous function of degree one.

Proof: Obvious. Q.E.D.

Corollary 14.2 Under Assumptions (F.1) through (F.7), a production function is homothetic in **X** and **Z** and weakly separable in **X** and $\mathbf{Z}$ if and only if the normalized restricted profit function is partially homothetic in $\mathbf{q}$ and $\mathbf{Z}$ and weakly separable in $\mathbf{q}$ and $\mathbf{Z}$.

Proof: By Lemma II-3, homotheticity and weak separability of $F(\mathbf{X},\mathbf{Z})$ in $\mathbf{X}$, $\mathbf{Z}$ implies that

$$Y = F(H_1(\mathbf{X}),H_2(\mathbf{Z})),$$

where $F(H_1,H_2)$ is homothetic in H_1 and H_2 and H_1 and H_2 are homogeneous functions of degree one. By Corollary 14.1,

$$\Pi^* = G(H_1^*(\mathbf{X}),H_2(\mathbf{Z})).$$

Partial homotheticity follows from Theorem II-13. The converse is proved similarly. Q.E.D.

Theorem II-15. Under Assumptions (F.1) through (F.7), a production function and its corresponding normalized restricted profit function are both separable in $\mathbf{X}$ and $\mathbf{q}$ respectively only if either they are homothetically separable in $\mathbf{X}$ and $\mathbf{q}$ respectively or they are additive.

Proof: The first-order necessary conditions for maximization imply that

$$\frac{\partial F/\partial X_i}{\partial F/\partial X_j} = \frac{\partial f/\partial X_i(X_1,\ldots,X_m)}{\partial f/\partial X_j(X_1,\ldots,X_m)} = \frac{q_i}{q_j}.$$

By a dual transformation this becomes

$$\frac{f_i(-\partial G/\partial q_1,\ldots,-\partial G/\partial q_m)}{f_j(-\partial G/\partial q_1,\ldots,-\partial G/\partial q_m)} = \frac{q_i}{q_j}.$$

Differentiating this equation with respect to Z_k, we obtain

$$f_j \sum_l f_{il}\left(-\frac{\partial^2 G}{\partial q_l \partial Z_k}\right) - f_i \sum_l f_{jl}\left(-\frac{\partial^2 G}{\partial q_l \partial Z_k}\right) = 0. \tag{II-11}$$

Since G is separable $[\Pi^* = G(g(\mathbf{q}),\mathbf{Z})]$,

$$\frac{\partial^2 G}{\partial q_l \partial Z_k} = \frac{\partial^2 G}{\partial g_l \partial Z_k}\cdot\frac{\partial g}{\partial q_l}.$$

We know $\mathbf{X} = -\partial G/\partial g \cdot \partial g/\partial \mathbf{q}$. Thus, equation (II-11) becomes

$$f_j \sum_l f_{il} X_l - f_i \sum_l f_{jl} X_l = 0,$$

or $f(\mathbf{X})$ is homothetic.

An exceptional case arises if $G_{gk} = 0, \forall k$. Then

$$G(\mathbf{q},\mathbf{Z}) = g(\mathbf{q}) + h(\mathbf{Z}),$$

and then by Theorem II-4,

$$F(\mathbf{X},\mathbf{Z}) = f(\mathbf{X}) + h(\mathbf{Z}). \qquad \text{Q.E.D.}$$

Additional results on the structure of normalized restricted profit functions can be found in Lau (1976a).

Based on these theorems, one can specify $G(\mathbf{q},\mathbf{Z})$ depending on the assumptions one wishes to impose on the underlying technology. As seen in Lau (1969c), it is generally difficult to obtain closed form solutions for the normalized profit function for even simple technologies when some inputs are fixed. With the device of the normalized restricted profit function, this problem of specification is circumvented. Nonetheless, we are assured that the resulting system of conditional supply and demand functions may be derived from an underlying neoclassical technology and that all the empirically relevant assumptions have been incorporated.

3. Extensions to Multiple Outputs

3.1. *Introduction*

A natural extension of the concept of profit functions is to the case of multiple outputs. This has been accomplished by McFadden (1966). Here we shall point out certain special properties of multiple output profit functions as well as derive several theorems on the structure of such functions. Our results may be further extended to include technologies in which the same commodity may be used either as a net input or a net output, depending on the market prices. Such technologies are not infrequently found. An example is the purchase and sale of farm-produced fertilizers by agricultural households. Further examples are those of international trade, and the purchase and sale of new and used equipment. The advantage of this approach is that there need be no arbitrary partition of commodities into inputs and outputs.

The theory of the multiproduct firm has been analyzed by Mundlak (1964). The properties of profit functions have been studied by McFadden (1966), Diewert (1973a) and Jorgenson and Lau (1974a and 1974b). Christensen, Jorgenson and Lau (1971 and 1973) have also made an empirical application to the U.S. economy. In addition, Hall (1973) has approached the problem from the point of view of joint cost functions, using a generalization of the Generalized Leontief cost function due to Diewert (1971). The basic duality concepts which underly all these studies may be traced back to the pioneering work of Shephard (1953).

For a multiple-output, multiple-input firm, there is no natural numeraire commodity, such as the single output, to define the production function representation of the technology. Following Jorgenson and Lau (1974a and 1974b), we shall adopt the convention of choosing as our left-hand-side variable for the production function a variable input which is non-producible. In addition, every commodity is measured as if it were a net output. Thus, a net output is always non-negative. A net input is always non-positive. For the purposes of this paper we maintain the artificial distinction between a set of commodities which are net outputs and the set of commodities which are net inputs.[15] A more general treatment should allow a commodity to be either a net output or a net input depending on the prices and fixed factors.[16]

Let X_{m+1} be the quantity of the left-hand-side variable and non-producible net input, Y_i the quantity of the ith net output, $i = 1,\ldots,n$, and X_i the quantity of the ith net input, $i = 1,\ldots,m$. By convention then $X_{m+1} \leqq 0, X_i \leqq 0, \forall i$, and $Y_i \geqq 0, \forall i$. The production function is given by

$$L \equiv -X_{m+1} = F(\mathbf{Y},\mathbf{X}),$$

the minimum value of L for given values of $\mathbf{Y}$ and $\mathbf{X}$ such that the production plan $(\mathbf{Y},\mathbf{X},-L)$ is feasible.

It is assumed that $F(\mathbf{Y},\mathbf{X})$ possesses certain properties, which parallel similar properties of the single-output case:

(F*.1) *Domain.* F is a finite, non-negative, real-valued function defined on $\bar{R}^n_+ \times \bar{R}^m_- \cdot F(\mathbf{0},\mathbf{0}) = 0$.

(F*.2) *Continuity.* F is continuous on $\bar{R}^n_+ \times \bar{R}^m_-$.

[15]This actually involves little loss in generality since the functional consequence of an output and an input being the same "commodity" is that they are two products whose prices are in fixed proportions in the market.

[16]See Chapter I.1 and Jorgenson and Lau (1974a and 1974b).

(F*.3) *Smoothness.* F is continuously differentiable on $R_+^n \times R_-^m$, and the Euclidean norm of the gradient of F with respect to $\mathbf{Y}$ and $\mathbf{X}$ is unbounded for any sequence of $\mathbf{Y}, \mathbf{X}$ in $R_+^n \times R_-^m$ converging to a boundary point of $\bar{R}_+^n \times \bar{R}_-^m$.

(F*.4) *Monotonicity.* F is non-decreasing on $\bar{R}_+^n \times \bar{R}_-^m$ and strictly increasing on $R_+^n \times R_-^m$.

(F*.5) *Convexity.* F is convex on $\bar{R}_+^n \times \bar{R}_-^m$ and locally strongly convex on $R_+^n \times R_-^m$.

(F*.6) *Twice Differentiability.* F is twice continuously differentiable on $R_+^n \times R_-^m$.

(F*.7) *Boundedness.*

$$\lim_{\lambda \to \infty} \frac{F(\lambda \mathbf{Y}, \lambda \mathbf{X})}{\lambda} = \infty, \qquad \forall \mathbf{Y}, \mathbf{X} \in \bar{R}_+^n \times \bar{R}_-^m, \mathbf{Y}, \mathbf{X} \neq \mathbf{0}.$$

We note that one consequence of our domain assumption is that for any given vector of net inputs $\mathbf{X}$, any vector of net outputs $\mathbf{Y}$ may be produced with an appropriate choice of L. In other words, any one of the net inputs $\mathbf{X}$ may be indefinitely substituted by L. This is admittedly a restrictive assumption. For example, the technology represented by

$$L = \left[\frac{1}{Y_1^2 + Y_2^2} + \frac{1}{X}\right]^{-1}$$

violates our domain assumption, for if $Y_1^2 + Y_2^2 > -X$, L is negative.[17] However, as indicated in Section 1.3, it is a relatively straightforward matter to introduce a more restrictive domain assumption and make corresponding changes in the assumption on G. We therefore maintain our domain assumption as it stands for the sake of simplicity of exposition.

The normalized profit function is given by

$$G(\mathbf{p}, \mathbf{q}) = \sup_{\mathbf{Y}, \mathbf{X}} \{\mathbf{p}'\mathbf{Y} + \mathbf{q}'\mathbf{X} - F(\mathbf{Y}, \mathbf{X}) | \mathbf{Y}, \mathbf{X} \in \bar{R}_+^n \times \bar{R}_-^m\},$$

where $\mathbf{p}$ and $\mathbf{q}$ are respectively the normalized prices of $\mathbf{Y}$ and $\mathbf{X}$ in terms of L. The corresponding properties of the normalized profit

[17]This example is due to Daniel McFadden.

function are:

(G*.1) *Domain.* G is a finite, positive, real-valued function defined on $R_+^n \times R_+^m$.

(G*.2) *Continuity.* G is continuous on $R_+^n \times R_+^m$.

(G*.3) *Smoothness.* G is continuously differentiable on $R_+^n \times R_+^m$, and the Euclidean norm of the gradient of G with respect to $\mathbf{p}$ and $\mathbf{q}$ is unbounded for any sequence of $\mathbf{p}$, $\mathbf{q}$ in $R_+^n \times R_+^m$ converging to a boundary point of $\bar{R}_+^n \times \bar{R}_+^m$.

(G*.4) *Monotonicity.* $G(\mathbf{p},\mathbf{q})$ is strictly increasing in $\mathbf{p}$ and strictly decreasing in $\mathbf{q}$ on $R_+^n \times R_+^m$.

(G*.5) *Convexity.* $G(\mathbf{p},\mathbf{q})$ is locally strongly convex on $R_+^n \times R_+^m$.

(G*.6) *Twice Differentiability.* $G(\mathbf{p},\mathbf{q})$ is twice continuously differentiable on $R_+^n \times R_+^m$.

(G*.7) *Boundedness.*

$$\lim_{\lambda \to \infty} \frac{G(\lambda \mathbf{p}, \lambda \mathbf{q})}{\lambda} = \infty, \qquad \forall \mathbf{p},\mathbf{q} \in R_+^n \times R_+^m.$$

It can be proved that Assumptions (F*.1) through (F*.7) imply Assumptions (G*.1) through (G*.7) and *vice versa*. The proof closely parallels the arguments used earlier in the single output case. We omit the proof. Properties of profit functions under more general conditions are derived in Chapter I.1.

As in the single output case, the Legendre transformation also holds in this case, with the following dual relationships:

$$\frac{\partial F}{\partial \mathbf{Y}} = \mathbf{p}, \qquad \frac{\partial G}{\partial \mathbf{p}} = \mathbf{Y},$$

$$\frac{\partial F}{\partial \mathbf{X}} = \mathbf{q}, \qquad \frac{\partial G}{\partial \mathbf{q}} = \mathbf{X}.$$

$$F + G = \mathbf{p}'\mathbf{Y} + \mathbf{q}'\mathbf{X}$$

$$= \frac{\partial F}{\partial \mathbf{Y}} \cdot \mathbf{Y} + \frac{\partial F}{\partial \mathbf{X}} \cdot \mathbf{X}, \qquad = \mathbf{p} \cdot \frac{\partial G}{\partial \mathbf{p}} + \mathbf{q} \cdot \frac{\partial G}{\partial \mathbf{q}}.$$

These dual relations may also be used in the study of relationships between classes of production functions and normalized profit functions.

3.2. *Homogeneity and Separability*

In the case of a multiple output and multiple input production function, the ordinary concept of homogeneity of the production function needs to be modified. Intuitively, we want to say that a production function is in some sense homogeneous of degree k if, when all net inputs are scaled by the same proportion λ, $\lambda > 0$, all net outputs are scaled by the same proportion λ^k. In other words, if

$$L = F(\mathbf{Y},\mathbf{X}),$$

then

$$\lambda L = F(\lambda^k \mathbf{Y},\lambda \mathbf{X}),$$

or

$$F(\lambda^k \mathbf{Y},\lambda \mathbf{X}) = \lambda F(\mathbf{Y},\mathbf{X}).$$

This corresponds precisely to the concept of almost homogeneity introduced in Section 2.2. The production function is almost homogeneous of degree 1 and k.[18]

Theorem III-1. Under Assumptions (F*.1) through (F*.7), a production function is almost homogeneous of degree 1 and k, $k < 1$, in outputs, if and only if the normalized profit function is homogeneous of degree $1/(1-k)$ in the normalized output prices.

Proof: By Lemma II-4 almost homogeneity implies

$$k \sum_{i=1}^{n} \frac{\partial F}{\partial Y_i} \cdot Y_i + \sum_{i=1}^{m} \frac{\partial F}{\partial X_i} \cdot X_i = F(\mathbf{Y},\mathbf{X}).$$

By a dual transformation, this equation becomes

$$k \sum_{i=1}^{n} p_i \frac{\partial G}{\partial p_i} + \sum_{i=1}^{m} q_i \frac{\partial G}{\partial q_i} = -G + \sum_{i=1}^{n} \frac{\partial G}{\partial p_i} \cdot p_i + \sum_{i=1}^{m} \frac{\partial G}{\partial q_i} \cdot q_i,$$

[18]Nothing requires that the scale effects be uniform. One may in fact have

$$F(\lambda^{k_1} Y_1, \lambda^{k_2} Y_2, \ldots, \lambda^{k_n} Y_n ; \lambda \mathbf{X}) = \lambda F(\mathbf{Y},\mathbf{X}).$$

This is a straightforward generalization of almost homogeneity.

or

$$\sum_{i=1}^{n} p_i \frac{\partial G}{\partial p_i} = \frac{1}{(1-k)} G.$$

The converse may be proved by retracing the steps. Q.E.D.

Corollary 1.1. Under Assumptions (F*.1) through (F*.7), and almost homogeneity of degrees 1 and k,

$$R^* = \frac{1}{(1-k)} G,$$

and

$$C^* = \frac{k}{(1-k)} G,$$

where R^* is the profit-maximizing normalized revenue and C^* is the profit-maximizing normalized cost.

Proof: This follows from the last equation in the proof of the theorem. Q.E.D.

Corollary 1.2. Under Assumptions (F*.1) through (F*.7) and almost homogeneity of degrees 1 and k, the derived supply functions of the outputs are homogeneous of degree $1/(1-k)$ in $\mathbf{p}$.

Proof: These follow from the properties of partial derivatives of homogeneous functions. Q.E.D.

Lemma III-1. Under Assumptions (G*.1) and (G*.5), the profit function,

$$\Pi(\mathbf{p}^*, \mathbf{q}^*, w) = wG(\mathbf{p}, \mathbf{q}),$$

is homogeneous of degree k in $\mathbf{p}^*$ if and only if it is homogeneous of degree $(1-k)$ in $\mathbf{q}^*$ and w.

Proof: It is well-known that $\Pi(\mathbf{p}^*, \mathbf{q}^*, w)$ is homogeneous of degree one in all prices. Hence

$$\sum_{i=1}^{n} p_i^* \frac{\partial \Pi}{\partial p_i^*} + \sum_{i=1}^{m} q_i^* \frac{\partial \Pi}{\partial q_i^*} + w \frac{\partial \Pi}{\partial w} = \Pi.$$

By hypothesis, Π is homogeneous of degree k in $\mathbf{p}^*$. Thus

$$\sum_{i=1}^{n} p_i^* \frac{\partial \Pi}{\partial p_i^*} = k\Pi = \Pi - \sum_{i=1}^{m} q_i^* \frac{\partial \Pi}{\partial q_i^*} - w \frac{\partial \Pi}{\partial w},$$

which simplifies to

$$\sum_{i=1}^{m} q_i^* \frac{\partial \Pi}{\partial q_i} + w \frac{\partial \Pi}{\partial w} = (1-k)\Pi.$$

The converse is proved similarly. Q.E.D.

Corollary 1.3. Under Assumptions (F*.1) through (F*.7), the production function is almost homogeneous of degree 1 and k if and only if the profit function is homogeneous of degree $-k/(1-k)$ in the input prices.

Proof: This follows directly from the theorem and Lemma III-1. Q.E.D.

With multiple outputs and inputs a technology is said to be separable in outputs and inputs if there exist functions $f(\cdot)$ and $g(\cdot)$ such that

$$f(\mathbf{Y}) - g(\mathbf{X},L) = 0.$$

In terms of our particular representation of the production function, it is equivalent to

$$L = F(f(\mathbf{Y}),\mathbf{X}).$$

We shall work with separability in this form.

A profit function, $\Pi(\mathbf{p}^*,\mathbf{q}^*,w)$ is said to be separable in outputs and inputs if it can be written in the form

$$\Pi(f(\mathbf{p}^*),g(\mathbf{q}^*,w)).$$

Lemma III-2. Under Assumptions (F*.1) through (F*.7), a separable production function is almost homogeneous of degree 1 and k if and only if it can be written in the form

$$L = H_1(H_2(\mathbf{Y}),\mathbf{X}),$$

where H_1 is a homogeneous function of degree one in H_2 and $\mathbf{X}$ and H_2 is a homogeneous function of degree $1/k$.

Proof: Almost homogeneity of $F(f(\mathbf{Y}),\mathbf{X})$ implies that

$$k\frac{\partial F}{\partial f}\sum_{i=1}^{n}\frac{\partial f}{\partial Y_i}\cdot Y_i+\sum_{i=1}^{m}\frac{\partial F}{\partial X_i}\cdot X_i=F.$$

This implies that

$$\sum_{i=1}^{n}\frac{\partial f}{\partial Y_i}\cdot Y_i=\frac{F(f,\mathbf{X})-\sum_{i=1}^{m}(\partial F/\partial X_i)(f,\mathbf{X})\cdot X_i}{k(\partial F/\partial f)(f,\mathbf{X})}.$$

But the left-hand side is a function of $\mathbf{Y}$ only. Thus one must have

$$\sum_{i=1}^{n}\frac{\partial f}{\partial Y_i}\cdot Y_i=g(f).$$

By Lemma II-5, f is homothetic. Without loss of generality, one may assume that f is homogeneous of degree $1/k$, making necessary accommodations in $F(f,\mathbf{X})$. Thus

$$\sum_{i=1}^{n}\frac{\partial f}{\partial Y_i}\cdot Y_i=\frac{1}{k}f.$$

Substituting this into the original differential equation we obtain

$$\frac{\partial F}{\partial f}\cdot f+\sum_{i=1}^{m}\frac{\partial F}{\partial X_i}\cdot X_i=F,$$

that is, F is homogeneous in f and $\mathbf{X}$. It may be verified immediately that

$$H_1(H_2(\lambda^k\mathbf{Y}),\lambda\mathbf{X})=\lambda H_1(H_2(\mathbf{Y}),\mathbf{X}).\qquad \text{Q.E.D.}$$

Lemma III-3. Under Assumptions (G*.1) through (G*.7), a profit function is weakly separable if and only if it can be written in the form

$$\Pi=H(H_1(\mathbf{p}^*),H_2(\mathbf{q}^*,w)),$$

where H, H_1, H_2 are all homogeneous functions of degree 1.

Proof: $\Pi(f(\mathbf{p}^*),g(\mathbf{q}^*,w))$ is homogeneous of degree one in $\mathbf{p}^*$, $\mathbf{q}^*$, w. Thus

$$\frac{\partial\Pi}{\partial f}\cdot\sum_{i=1}^{n}\frac{\partial f}{\partial p_i^*}+\frac{\partial\Pi}{\partial g}\left(\sum_{i=1}^{m}\frac{\partial g}{\partial q_i^*}q_i^*+\frac{\partial g}{\partial w}\cdot w\right)=\Pi,$$

or

$$\sum_{i=1}^{n} \frac{\partial f}{\partial p_i^*} p_i^* = \frac{\Pi(f,g) - (\partial \Pi / \partial g)\left(\sum_{i=1}^{m} (\partial g / \partial q_i^*) q_i^* + (\partial g / \partial w) \cdot w\right)}{(\partial \Pi / \partial f)}.$$

Hence

$$\sum_{i=1}^{n} \frac{\partial f}{\partial p_i^*} p_i^* = h_1(f),$$

and

$$\sum_{i=1}^{m} \frac{\partial g}{\partial q_i^*} q_i^* + \frac{\partial g}{\partial w} \cdot w = h_2(g).$$

And $\Pi(f(\mathbf{p}^*),g(\mathbf{q}^*,w))$ may be chosen so that it is a homogeneous function of degree one of two homogeneous functions of degree one. Q.E.D.

Corollary 3.1. Under Assumptions (G*.1) through (G*.7), a profit function is weakly separable if and only if the normalized profit function can be written in the form

$$G = H(H_1(\mathbf{p}), g(\mathbf{q})),$$

where H and H_1 are homogeneous functions of degree one.

Proof:

$$\begin{aligned} G &= \frac{H(H_1(\mathbf{p}^*), H_2(\mathbf{q}^*, w))}{w} \\ &= H\left(\frac{H_1(\mathbf{p}^*)}{w}, \frac{H_2(\mathbf{q}^*, w)}{w}\right) \\ &= H(H_1(\mathbf{p}), H_2(\mathbf{q}, 1)) \\ &= H(H_1(\mathbf{p}), g(\mathbf{q})). \end{aligned}$$

The converse is obvious. Q.E.D.

We refer to such a normalized profit function as separable.

Corollary 3.2. Under the Assumptions (G*.1) through (G*.7), a profit function is separable in the input prices if and only if the normalized profit function can be written in the form

$$G = H(\mathbf{p}, f(\mathbf{q})),$$

where H is a homogeneous function of degree one in $\mathbf{p}$ and f.

Proof: We apply Lemma III-3, treating the price of each output as a separate group. Q.E.D.

Separability of the profit function implies that the optimal output proportions are independent of input prices and *vice versa*.

> *Theorem III*-2. Under Assumptions (F*.1) through (F*.7), a production function is almost homogeneous of degree 1 and k and separable if and only if the normalized profit function is homogeneous of degree $1/(1-k)$ in $\mathbf{p}$ and separable.

Proof: Almost homogeneity is equivalent to homogeneity by Theorem III-1. By Lemma III-2,

$$\frac{\partial L/\partial Y_i}{\partial L/\partial Y_j} = \frac{(\partial H_2/\partial Y_i)(\mathbf{Y})}{(\partial H_2/\partial Y_j)(\mathbf{Y})} = \frac{p_i}{p_j},$$

where H_2 is homogeneous of degree $1/k$. Then by the now familiar argument, $(\partial G/\partial p_i)/(\partial G/\partial p_j)$ is independent of q_k, $\forall k$.

Also by Lemma III-2,

$$\frac{\partial L}{\partial X_i} = \frac{\partial H_1}{\partial X_i}(H_2,\mathbf{X}) = q_i.$$

But H_1 is homogeneous of degree one which implies that $\partial H_1/\partial X_i$ is homogeneous of degree zero. Thus,

$$\frac{\partial H_1}{\partial X_i}\left(1, \frac{X_1}{H_2},\ldots,\frac{X_m}{H_2}\right) = q_i, \qquad i = 1,\ldots,m.$$

X_i/H_2 may be solved as functions of q alone. Hence X_i/X_j or $(\partial G/\partial q_i)/(\partial G/\partial q_j)$ is independent of p. Thus, we have shown that $\Pi^* = G(f(\mathbf{p}),g(\mathbf{q}))$, where G is in addition homogeneous of degree $1/(1-k)$ in $\mathbf{p}$. By Euler's Theorem,

$$\frac{\partial G}{\partial f}\cdot\sum_{i=1}^{n}\frac{\partial f}{\partial p_i}p_i = \frac{1}{(1-k)}G.$$

By the usual argument, one can choose

$$\sum_{i=1}^{n}\frac{\partial f}{\partial p_i}p_i = \frac{1}{(1-k)}G.$$

Therefore,

$$\frac{\partial \ln G}{\partial \ln f} = 1,$$

or

$$\ln G = \ln f + h(g).$$

Thus, one has

$$G = H_1(\mathbf{p})^{1/(1-k)} g^*(\mathbf{q}),$$

and hence G is separable.

To prove the converse, note that separability of the normalized profit function implies, by Lemma III-3, that it has the form

$$G = H(H_1(\mathbf{p}), g(\mathbf{q})).$$

Homogeneity of G of degree $1/(1-k)$ in $\mathbf{p}$ implies that

$$\frac{\partial G}{\partial H_1} \cdot H_1 = \frac{1}{(1-k)} G.$$

Hence

$$G(\mathbf{p},\mathbf{q}) = H_1(\mathbf{p})^{1/(1-k)} g^*(\mathbf{q}).$$

It then follows from homogeneity of $H_1(\mathbf{p})$ by the usual argument that $(\partial F/\partial Y_i)/(\partial F/\partial Y_j)$ is independent of $\mathbf{X}$. Almost homogeneity of F follows from homogeneity of $G(\mathbf{p},\mathbf{q})$ in $\mathbf{p}$. Q.E.D.

Corollary 2.1. Under Assumptions (F*.1) through (F*.7), a production function is almost homogeneous and separable if and only if the profit function can be written in the form

$$\Pi(\mathbf{p}^*, \mathbf{q}^*, w) = H_1(\mathbf{p}^*)^{1/(1-k)} H_2(\mathbf{q}^*, w)^{-k/(1-k)}.$$

Proof: This result is obtained by straightforward substitution. Q.E.D.

Theorem III-3. Under Assumptions (F*.1) through (F*.7), a production function is homothetically separable in outputs if and only if the normalized profit function is homothetically separable in output prices.

Proof: By hypothesis, $L = F(H(\mathbf{Y}),\mathbf{X})$. Thus,

$$\frac{\partial H/\partial Y_i}{\partial H/\partial Y_j} = \frac{p_i}{p_j}.$$

Hence the optimal output ratios, $(\partial G/\partial p_i)/(\partial G/\partial p_j)$ may be solved in terms of (p_i/p_j)'s alone. Moreover, they are homogeneous of degree zero in $\mathbf{p}$. The normalized profit function is therefore homothetically separable in output prices. The converse is similarly proved. Q.E.D.

Theorem III-4. Under Assumptions (F*.1) through (F*.7), a production function is homothetically separable in inputs if and only if the normalized profit function is separable in input prices, that is, has the form

$$G = g(\mathbf{q})g^*\left(\frac{\mathbf{p}}{g(\mathbf{q})}\right).$$

Proof: Homothetic separability in inputs implies that

$$L = F(\mathbf{Y},\mathbf{X}) = H(f(\mathbf{Y}),\mathbf{X}),$$

where H is homogeneous of degree one in $f(\mathbf{Y})$ and $\mathbf{X}$. This can be alternatively written as

$$L = f(\mathbf{Y})g\left(\frac{\mathbf{X}}{f(\mathbf{Y})}\right).$$

The first-order necessary conditions for a maximum are

$$\frac{\partial L}{\partial X_i} = g_i\left(\frac{X}{f(\mathbf{Y})}\right) = q_i, \qquad i = 1,\ldots,m.$$

Thus, one may solve $X_i/f(\mathbf{Y})$ as unique and continuously differentiable functions of the $\mathbf{q}$ alone. This implies

$$X_i = g_i^*(\mathbf{q})f(\mathbf{Y}).$$

Substituting this into the production function we have

$$\begin{aligned} \Pi^* &= \max_{\mathbf{Y}}\{p'\mathbf{Y} + f(\mathbf{Y})g(\mathbf{q})\} \\ &= g(\mathbf{q})g^*\left(\frac{\mathbf{p}}{g(\mathbf{q})}\right), \end{aligned}$$

By Theorem II-3. The converse is proved similarly. Q.E.D.

Theorem III-5. Under Assumptions (F*.1) through (F*.7), a production function is homothetically separable in both outputs and inputs if and only if

$$G(\mathbf{p},\mathbf{q}) = g(\mathbf{q})g^*\left(\frac{H(\mathbf{p})}{g(\mathbf{q})}\right).$$

where H is a homogeneous function of degree one.

Proof: This theorem follows directly from the two previous theorems. Q.E.D.

Theorem III-6. Under Assumption (F*.1) through (F*.7), a production function and its corresponding normalized profit function are both separable in outputs only if either they are homothetically separable in outputs or they are additive.

Proof:

$$F(\mathbf{Y},\mathbf{X}) = F(f(\mathbf{Y}),\mathbf{X}),$$
$$G(\mathbf{p},\mathbf{q}) = G(g(\mathbf{p}),\mathbf{q}).$$

The first-order necessary conditions for a maximum imply

$$\frac{f_i(Y_1,\ldots,Y_n)}{f_j(Y_1,\ldots,Y_n} = \frac{p_i}{p_j}.$$

By a dual transformation, this becomes

$$\frac{f_i(\partial G/\partial p_1,\ldots,\partial G/\partial p_n)}{f_j(\partial G/\partial p_1,\ldots,\partial G/\partial p_n)} = \frac{p_i}{p_j}.$$

Differentiating these equations with respect to q_k, we obtain

$$f_i \sum_{l=1}^{n} f_{jl}G_{lk} - f_j \sum_{l=1}^{n} f_{il}G_{lk} = 0.$$

But

$$G_{lk} = G_{gk}g_l, \qquad G_l = G_g g_l = Y_l.$$

Hence

$$f_i \sum_{l=1}^{n} f_{jl}Y_l - f_j \sum_{l=1}^{n} f_{il}Y_l = 0,$$

which means f_i/f_j is homogeneous of degree zero in $\mathbf{Y}$. Hence f is homothetic in $\mathbf{Y}$ by Lemma II-1. It follows from Theorem II-9 that $g(\mathbf{p})$ is also homothetic.

An exceptional case arises if $G_{gk} = 0$, $\forall k$. Then

$$G(\mathbf{p},\mathbf{q}) = g(\mathbf{p}) + h^*(\mathbf{q}).$$

And by Theorem II-7,

$$F(\mathbf{Y},\mathbf{X}) = f(\mathbf{Y}) + h(\mathbf{X}). \qquad \text{Q.E.D.}$$

Extension to the case with fixed inputs is straightforward and will not be repeated here.

3.3. *Non-Jointness in Production*

The problem of non-jointness has been investigated by Samuelson (1966) who derives necessary and sufficient conditions for a production function to represent a non-joint technology. Hall (1973) has approached the problem using the joint cost function. It turns out that the assumption of non-jointness of the technology implies very simple restrictions on the matrix of second partial derivatives of the normalized profit function. We shall present these results. First of all we give a definition.

Definition. A production function $L = F(\mathbf{Y},\mathbf{X})$ is said to be *non-joint in inputs* if there exist individual production functions,

$$L_i = f_i(Y_i,X_{1i},X_{2i},\dots,X_{mi}), \qquad i = 1,\dots,n.$$

such that

$$F(\mathbf{Y},\mathbf{X}) = \min\left\{\sum_{i=1}^{n} f_i(Y_i,X_{1i},X_{2i},\dots,X_{mi}) \middle| \sum_{i=1}^{n} X_{ji} \geqq X_j, \quad j = 1,\dots,m\right\}.$$

A production function is said to be *non-joint in outputs* if there exists individual production functions,

$$L = g_0(Y_{10},\dots,Y_{n0}),$$
$$X_i = g_i(Y_{1i},\dots,Y_{ni}), \qquad i = 1,\dots,m,$$

such that

$$F(\mathbf{Y},\mathbf{X}) = \min\left\{g_0(Y_{10},\dots,Y_{n0}) | X_i \leqq g_i(Y_{1i},\dots,Y_{ni}), \qquad i = 1,\dots,m, \right.$$
$$\left. \sum_{i=0}^{m} Y_{ji} \geqq Y_j, \quad j = 1,\dots,n\right\}.$$

The minimum in these two definitions ensure that all the inputs (and outputs) are allocated amongst the individual industries so that production is efficient, that is, the output of no one industry may be increased without decreasing the output of another industry. (And no one input may be decreased without increasing another input.)

The normalized profit function of a technology characterized by non-jointness in inputs has a very simple representation: it is the sum of the individual normalized profit functions corresponding to the individual industry production functions. This is embodied in the following theorem:

Theorem III-7. Under Assumptions (F*.1) through (F*.7), a production function is non-joint in inputs if and only if its normalized profit function is additive in $\mathbf{p}$, that is,

$$\Pi^* = \sum_{i=1}^{n} G_i(p_i,\mathbf{q}).$$

Proof:
Necessity.

$$\begin{aligned}\Pi^* &= \max_{Y_i,X_{ij}} \left\{\sum_{i=1}^{n} p_i Y_i + \sum_{j=1}^{m} q_j \sum_{i=1}^{n} X_{ji} - \sum_{i=1}^{n} f_i(Y_i,X_{1i},\ldots,X_{mi})\right\} \\ &= \sum_{i=1}^{n} \max_{Y_i,X_{ij}} \left\{p_i Y_i + \sum_{j=1}^{m} q_j X_{ji} - f_i(Y_i,X_{1i},\ldots,X_{mi})\right\} \\ &= \sum_{i=1}^{n} G_i(p_i,\mathbf{q}).\end{aligned}$$

Sufficiency. Given $\Pi^* = \sum_{i=1}^{n} G_i(p_i,\mathbf{q})$, one can find for each $G_i(p_i,\mathbf{q})$ a unique production function $L_i = f_i(Y_i,X_{1i},\ldots,X_{mi})$. Thus, the technology is non-joint in inputs. Q.E.D.

Corollary 7.1. Under Assumptions (F*.1) through (F*.7), a production function is non-joint in inputs if and only if

$$\frac{\partial^2 G}{\partial p_i \partial p_j} = 0, \qquad i \neq j, \quad \forall i,j,$$

where G is the normalized profit function and $\mathbf{p}$ is the vector of normalized output prices.[19]

Proof: This follows directly from the theorem. Q.E.D.

[19]This condition is also given by Diewert (1973a).

Note that duality of $\sum_{i=1}^{n} G_i(p_i,\mathbf{q})$ to a non-joint in inputs technology is an immediate consequence of the convolution theorem for profit functions in Chapter I.1.

Corollary 7.1 provides a very useful necessary and sufficient condition for the characterization of a "non-joint in inputs" technology. In particular, it lends itself to straightforward empirical tests. In retrospect: it turns out that our conditions here are completely equivalent to the conditions stated by Samuelson (1966) on the Hessian of the production function. One needs only recall from Section 1 that the Hessian matrix of the normalized profit function $G(\mathbf{p},\mathbf{q})$ is the inverse of the Hessian matrix of $F(\mathbf{Y},\mathbf{X})$. Hence singularity conditions on the minors of the Hessian matrix of F are equivalent to zero conditions on the elements of the Hessian of $G(\mathbf{p},\mathbf{q})$.

On the other hand, in the case of non-jointness in outputs, it is easy to see that the normalized profit function is given by

$$\Pi^* = G_0(\mathbf{p}) + \sum_{i=1}^{m} q_i G_i\left(\frac{\mathbf{p}}{q_i}\right).$$

Theorem III-8. Under Assumptions (F*.1) through (F*.7), a production function is non-joint in outputs if and only if its normalized profit function can be written in the form

$$\Pi^* = G_0(\mathbf{p}) + \sum_{i=1}^{m} q_i G_i\left(\frac{\mathbf{p}}{q_i}\right).$$

Proof: Obvious. Q.E.D.

3.4. *Summary*

We may summarize the results of Sections 3.2 and 3.3 by way of a table which describes the restrictions on the normalized profit functions under alternative combinations of assumptions on the technology. The alternatives considered are as follows:[20]

(1) almost homogeneity of the production function,
(2) direct separability,

[20]This table is different from that of Lau (1972) in two respects: first, the forms are specified in terms of the normalized profit function; second, some of the errors have been corrected and "open" questions have been closed.

TABLE 1
Functional forms of normalized profit functions under alternative assumptions.

(1)	$H_p^k(\mathbf{p},\mathbf{q})$
(2)	$\sup_{\lambda}\{H_p(\lambda,\mathbf{p})-G(\lambda,\mathbf{q})\}$
(3)	$G(\mathbf{q})G^*\left(\frac{H(\mathbf{p})}{G(\mathbf{q})}\right)$
(4)	$\sum_{i=1}^{n} G_i(p_i,\mathbf{q})$
(5)	$G_0(\mathbf{p})+\sum_{i=1}^{m} q_iG_i\left(\frac{\mathbf{p}}{q_i}\right)$
(1) + (2)	$H^k(\mathbf{p})G(\mathbf{q})$
(1) + (3)	$H^k(\mathbf{p})G(\mathbf{q})$
(1) + (4)	$\sum_{i=1}^{n} p_i^kG_i(\mathbf{q})$
(1) + (5)	$H_0^k(\mathbf{p})+\sum_{i=1}^{m} q_iH_i^k\left(\frac{\mathbf{p}}{q_i}\right)$
(2) + (3)	Same as (3)
(2) + (4)	$F(\mathbf{q})\sum_{i=1}^{n} G_i\left(\frac{p_i}{F(\mathbf{q})}\right)+G(\mathbf{q})$
(2) + (5)	$G_0(H(\mathbf{p}))+\sum_{i=1}^{m} q_iG_i\left(\frac{H(\mathbf{p})}{q_i}\right)+H^*(\mathbf{p})$
(3) + (4)[a]	$\left[\sum_{i=1}^{n}\alpha_ip_i^k\right]G(\mathbf{q})^{1-k}+G(\mathbf{q})$
(3) + (5)[b]	$\left[\alpha_0+\sum_{i=1}^{m}\alpha_iq_i^k\right]H(\mathbf{p})^{1-k}+H(\mathbf{p})$ or $\left[\alpha_0+\sum_{i=1}^{m}\alpha_i\ln\left(\frac{q_i}{H(\mathbf{p})}\right)\right]H(\mathbf{p})$
(4) + (5)	$\sum_{i=1}^{n}\left[g_{i0}(p_i)+\sum_{j=1}^{m} q_jg_{ij}^*\left(\frac{p_i}{q_j}\right)\right]$
(1) + (2) + (3)	$H^k(\mathbf{p})G(\mathbf{q})$
(1) + (2) + (4)	Same as (3) + (4)
(1) + (2) + (5)	Same as (3) + (5)
(1) + (3) + (4)	Same as (3) + (4)
(1) + (3) + (5)	Same as (3) + (5)
(3) + (4) + (5)[c]	$\left[\sum_{i=1}^{n}\alpha_ip_i\right]+\left[\beta_0+\sum_{i=1}^{m}\beta_iq_i\right]$

[a]This implies that the production functions for all outputs are identical up to a multiplicative constant. Both Denny (1972) and Hall (1973) have independently discovered this result in the context of joint cost functions.

[b]Likewise, this implies that all the production functions are identical up to a multiplicative constant.

[c]This implies a fixed-coefficients type technology. Note that this violates our local strong convexity assumption. In principle,

$$\left[\alpha_0+\sum_{i=1}^{m}\alpha_iq_i^{1-k}\right]\left[\sum_{i=1}^{n}\beta_ip_i^k\right]$$

is a possible solution. However, this solution cannot satisfy the monotonicity and convexity conditions simultaneously.

(3) indirect separability,
(4) non-jointness in inputs,
(5) non-jointness in outputs.

$F(\cdot)$ and $G(\cdot)$ are used to denote arbitrary functions: $H(\cdot)$, $H^*(\cdot)$ and $H^{**}(\cdot)$ are used to denote arbitrary homogeneous functions; and $H'(\cdot)$ is used to denote arbitrary homothetic functions. A subscript denotes the set of variables in which the function is homogeneous or homothetic. A superscript denotes the degree of homogeneity when it is different from one.

Many of these combinations are obvious. We shall derive three of the relatively less obvious ones.

3.4.1. *Derivation of* (2) + (4) *and* (2) + (5)

Direct separability implies that the normalized cost function of producing $f(\mathbf{Y})$ can be written as

$$C^* = G(f(\mathbf{Y}),\mathbf{q}). \tag{III-1}$$

Non-jointness in inputs implies that the normalized cost functions can be written as

$$C^* = \sum_{i=1}^{n} f_i(Y_i,\mathbf{q}). \tag{III-2}$$

We note that C^* is characterized by $(\partial^2 C^*)/(\partial Y_i \partial Y_j) = 0$, $i \neq j$. Differentiating equation (III-1), we obtain

$$\frac{\partial^2 C^*}{\partial Y_i \partial Y_j} = \frac{\partial G}{\partial f}\cdot\frac{\partial^2 f}{\partial Y_i \partial Y_j} + \frac{\partial^2 G}{\partial f^2}\cdot\frac{\partial f}{\partial Y_i}\frac{\partial f}{\partial Y_j} = 0, \qquad i \neq j,$$

which implies

$$\frac{\partial^2 G/\partial f^2}{\partial G/\partial f} = -\frac{(\partial^2 f)/(\partial Y_i \partial Y_j)}{(\partial f/\partial Y_i)/(\partial f/\partial Y_j)}, \quad \text{if} \quad \frac{\partial^2 G}{\partial f^2} \neq 0,$$

or

$$\frac{\partial^2 f}{\partial Y_i \partial Y_j} = 0, \qquad\qquad \text{if} \quad \frac{\partial^2 G}{\partial f^2} = 0.$$

We note that in the first case the right-hand side of the equation is independent of $\mathbf{q}$. Hence the left-hand side is independent of $\mathbf{q}$. Further we observe that the left-hand side may be written as $(\partial/\partial f)\ln(\partial G/\partial f)$

which is a function of f alone. Thus, by successive integration we obtain

$$G(f,\mathbf{q}) = g(f)h_1(\mathbf{q}) + h_2(\mathbf{q}),$$

which becomes, in order to satisfy equation (III-2),

$$C^* = \sum_{i=1}^{n} g_i(Y_i)h_1(\mathbf{q}) + h_2(\mathbf{q}).$$

Note that this implies that the isoquants of each industry have the same shape although the numberings may differ.

Now the normalized profit function is given by

$$\begin{aligned} \Pi^* &= \max_{\mathbf{Y}} \left\{\sum_{i=1}^{n} p_i Y_i - \sum_{i=1}^{n} g_i(Y_i)h_1(\mathbf{q}) - h_2(\mathbf{q})\right\} \\ &= \sum_{i=1}^{n} \max_{Y_i}\{p_i Y_i - g_i(Y_i)h_1(\mathbf{q})\} - h_2(\mathbf{q}) \\ &= h_1(\mathbf{q}) \sum_{i=1}^{n} \max_{Y_i} \left\{\frac{p_i}{h_1(\mathbf{q})} Y_i - g_i(Y_i)\right\} - h_2(\mathbf{q}) \\ &= h_1(\mathbf{q}) \sum_{i=1}^{n} g_i^*\left(\frac{p_i}{h_1(\mathbf{q})}\right) - h_2(\mathbf{q}).^{21} \end{aligned}$$

In the second case, we also have

$$C^* = \sum_{i=1}^{n} f_i(Y_i)h_1(\mathbf{q}) + h_2(\mathbf{q}).$$

The condition for (2) + (5) may be derived similarly.

3.4.2. *Derivation of* (3) + (4) *and* (3) + (5)

Indirect separability implies that the normalized profit function can be written as

$$\Pi^* = H(H^*(\mathbf{p}), f(\mathbf{q})) = f(\mathbf{q})G\left(\frac{H^*(\mathbf{p})}{f(\mathbf{q})}\right).$$

Non-jointness in inputs implies that the normalized profit function can be written as

$$\Pi^* = \sum_{i=1}^{n} F_i(p_i,\mathbf{q}).$$

[21]This is precisely the form suggested by Professor W. M. Gorman to the author in 1970. At that time the author was unable to establish the necessity of this form. See Lau (1972, p. 288, fn. 20).

We note that Π^* is characterized by $(\partial^2\Pi^*)/(\partial p_i \partial p_j) = 0$, $i \neq j$. This implies that

$$\frac{G'' H_i^* H_j^*}{f} + G' H_{ij}^* = 0, \qquad G'' \neq 0,$$

or

$$\frac{G''(H^*/f)}{G'} + \frac{H_{ij}^* H^*}{H_i^* H_j^*} = 0, \qquad G'' \neq 0, \qquad \text{(III-3)}$$

and

$$H_{ij}^* = 0, \quad i \neq j, \qquad G'' = 0.$$

The second term of equation (III-3) is independent of f, which implies that the first term is independent of f. But the first term being independent of f means it is independent of H^*/f, since $G(\cdot)$ is a function of a single variable H^*/f only. Hence the first term must be constant, that is

$$\frac{G''}{G'}(H^*/f) = k,$$

k a constant. This equation may be integrated to yield,

$$G(Z) = C_1 \frac{Z^{k+1}}{k+1} + C_2, \qquad k \neq -1,$$
$$= C_1 \ln Z + C_2, \qquad k = -1.$$

For $k \neq -1$, one must have

$$\frac{C_1 H^*(\mathbf{p})^{k+1}}{k+1}$$

be additive and homogeneous of degree $k+1$ in $\mathbf{p}$. This means

$$C_1 H^*(\mathbf{p})^{k+1} = \sum_{i=1}^{n} \alpha_i p_i^{k+1}.$$

For $k = -1$, one must have

$$C_1 \ln H^*(\mathbf{p}) + C_2$$

be additive and homothetic, which implies

$$C_1 \ln H^*(\mathbf{p}) = \sum_{i=1}^{n} \alpha_i \ln p_i.$$

Note, however, that this is inconsistent with the monotonicity and convexity requirements of the normalized profit function. Thus, the only possibility is that of

$$\Pi^* = f(q)\left[\alpha_0 + \sum_{i=1}^{n} \alpha_i \left(\frac{p_i}{f(\mathbf{q})}\right)^{k+1}\right].$$

If in addition we require that $\Pi^* = 0$ if $p_i = 0, \forall i$, then

$$\Pi^* = \left[\sum_{i=1}^{n} \alpha_i p_i^{k^*}\right] f(\mathbf{q})^{1-k^*}.$$

Alternatively, if $G'' = 0$, $H_{ij}^* = 0$, which implies that the normalized profit function must have the form

$$\begin{aligned}\Pi^* &= f(\mathbf{q})\left[\alpha_0 + \frac{\sum_{i=1}^{n} \alpha_i p_i}{f(\mathbf{q})}\right] \\ &= \sum_{i=1}^{n} \alpha_i p_i + f(\mathbf{q})\alpha_0.\end{aligned}$$

The condition for (3) + (5) may be derived similarly.

3.4.3. *Derivation of* (4) + (5)

$$\Pi^* = \sum_{i=1}^{n} G_i(p_i, \mathbf{q}) \tag{III-4}$$

$$= G_0(\mathbf{p}) + \sum_{i=1}^{m} q_i G_i\left(\frac{\mathbf{p}}{q_i}\right). \tag{III-5}$$

Since $p_i = 0$ implies $G_i = 0$ in equation (III-4), then $(\partial^2 \Pi^*)/(\partial q_i \partial q_j) = 0$ implies that each G_i must have the form

$$g_{i0}(p_i) + \sum_{j=1}^{m} g_{ij}(p_i, q_j).$$

Substituting this into equation (III-4) leads to

$$\Pi^* = \sum_{i=1}^{n} \left[g_{i0}(p_i) + \sum_{j=1}^{m} q_j g_{ij}^*\left(\frac{p_i}{q_j}\right)\right].$$

4. Examples of Normalized Profit Functions

4.1. *Introduction*

In this section we present a number of examples of normalized profit functions. In particular, we demonstrate how the theorems derived in Sections 2 and 3 and the Legendre transformation may be used in the construction of the normalized profit function given the production function (and *vice versa*).

4.2. *Cobb–Douglas Production Function*

Let

$$Y = \prod_{i=1}^{m} X_i^{\alpha_i}.$$

The first-order necessary conditions for a maximum are

$$\frac{\alpha_i Y}{X_i} = q_i, \qquad i = 1,\ldots,m. \qquad \text{(IV-1)}$$

By Theorem (II-1), $Y = (1-\mu)^{-1}G$ where $\mu = \sum_{i=1}^{m} \alpha_i (<1)$ because Y is homogeneous of degree μ in $\mathbf{X}$. Hence, by a dual transformation, equation (IV-1) becomes

$$\frac{\alpha_i (1-\mu)^{-1} G}{-\partial G/\partial q_i} = q_i, \qquad i = 1,\ldots,m,$$

which may be integrated as

$$G(\mathbf{q}) = A^* \prod_{i=1}^{m} q_i^{\alpha_i^*},$$

where $\alpha_i^* = -\alpha_i(1-\mu)^{-1}$, $i = 1,\ldots,m$, and A^* is a constant of integration. A^* may be determined from initial conditions. For instance, at $q_i = 1$, $i = 1,\ldots,m$, equation (IV-1) implies that $X_i = \alpha_i Y$, $i = 1,\ldots,m$. Substituting this into the production function we have

$$Y = \prod_{i=1}^{m} X_i^{\alpha_i} = \prod_{i=1}^{m} \alpha_i^{\alpha_i} Y^{\alpha_i}.$$

Therefore,

$$\begin{aligned} Y &= \left(\prod_{i=1}^{m} \alpha_i^{\alpha_i}\right)^{1/(1-\mu)} \\ &= (1-\mu)^{-1} G(1) \\ &= (1-\mu)^{-1} A^*. \end{aligned}$$

Thus,

$$A^* = (1-\mu) \prod_{i=1}^{m} \alpha_i^{\alpha_i(1-\mu)^{-1}},$$

and

$$G(\mathbf{q}) = (1-\mu) \prod_{i=1}^{m} \left(\frac{q_i}{\alpha_i}\right)^{-\alpha_i(1-\mu)^{-1}}$$

To extend this to the case with fixed inputs, we have

$$Y = \prod_{i=1}^{m} X_i^{\alpha_i} \prod_{i=1}^{n} Z_i^{\beta_i}.$$

Then, by applying Theorem (II-3), one has immediately,

$$\begin{aligned} G(\mathbf{q},\mathbf{Z}) &= \prod_{i=1}^{n} Z_i^{\beta_i} (1-\mu) \prod_{i=1}^{m} \left(\frac{q_i}{\alpha_i \prod_{i=1}^{n} Z_i^{\beta}}\right)^{-\alpha_i(1-\mu)^{-1}} \\ &= (1-\mu) \prod_{i=1}^{m} \left(\frac{q_i}{\alpha_i}\right)^{-\alpha_i(1-\mu)^{-1}} \prod_{i=1}^{n} Z_i^{\beta_i(1-\mu)^{-1}} \end{aligned}$$

For this latter case, the supply function is, again by Theorem (II-1),

$$\begin{aligned} Y(\mathbf{q},\mathbf{Z}) &= (1-\mu)^{-1} G \\ &= \prod_{i=1}^{m} \left(\frac{q_i}{\alpha_i}\right)^{-\alpha_i(1-\mu)^{-1}} \prod_{i=1}^{n} Z_i^{\beta_i(1-\mu)^{-1}}, \end{aligned}$$

and the derived demand functions are

$$\begin{aligned} X_j &= -\frac{\partial G}{\partial q_j} \\ &= \frac{\alpha_j}{q_j}\left[\prod_{i=1}^{m} \left(\frac{q_i}{\alpha_i}\right)^{-\alpha_i(1-\mu)^{-1}} \prod_{i=1}^{n} Z_i^{\beta_i(1-\mu)^{-1}}\right], \qquad j = 1,\ldots,m. \end{aligned}$$

We note also that for the Cobb–Douglas production function the expenditure on each variable input is a constant proportion of profits. This follows from

$$\frac{q_i X_i}{G} = -\frac{\partial \ln G}{\partial \ln q_i} = \alpha_i (1-\mu)^{-1}, \qquad i = 1,\ldots,m.$$

4.3. *The C.E.S. Production Function*

We have

$$Y = \left[\sum_{i=1}^{m} \alpha_i X_i^{\rho}\right]^{\mu/\rho},$$

where $\mu < 1$ is a scale parameter.

The necessary conditions for a maximum are

$$\frac{\partial Y}{\partial X_j} = \mu \left[\sum_{i=1}^{m} \alpha_i X_i^{\rho}\right]^{(\mu-\rho)/\rho} \alpha_j X_j^{\rho-1}$$
$$= \mu Y^{(\mu-\rho)/\mu} \alpha_j X_j^{\rho-1} = q_j, \qquad j = 1,\ldots,m,$$

which may be rewritten as

$$-\mu^{1/(\rho-1)}[(1-\mu)^{-1}G]^{(\mu-\rho)/\mu(\rho-1)} \alpha_j^{1/(\rho-1)} \frac{\partial G}{\partial q_j} = q_j^{1/(\rho-1)}, \qquad j = 1,\ldots,m,$$

which becomes

$$-\mu^{1/(\rho-1)}(1-\mu)^{-(\mu-\rho)/\mu(\rho-1)} \cdot \frac{\mu(\rho-1)}{p(\mu-1)} \frac{\partial G^{\rho(\mu-1)/\mu(\rho-1)}}{\partial q_j} = \left(\frac{q_j}{\alpha_j}\right)^{1/(\rho-1)},$$

which may be integrated as[22]

$$\mu^{\rho/(\rho-1)}(1-\mu)^{\rho(1-\mu)/\mu(\rho-1)} \cdot G^{-\rho(1-\mu)/\mu(\rho-1)} = \sum_{i=1}^{m} \alpha_i \left(\frac{q_i}{\alpha_i}\right)^{\rho/(\rho-1)}.$$

Hence

$$G(\mathbf{q}) = \mu^{\mu(1-\mu)^{-1}}(1-\mu)\left[\sum_{i=1}^{m} \alpha_i \left(\frac{q_i}{\alpha_i}\right)^{\rho/(\rho-1)}\right]^{-(\mu(1-\mu)^{-1}(\rho-1))/\rho}.$$

By Theorem (II-1) the supply function is immediately given by[23]

$$Y = (1-\mu)^{-1}G(\mathbf{q})$$
$$= \mu^{\mu(1-\mu)^{-1}}\left[\sum_{i=1}^{m} \alpha_i \left(\frac{q_i}{\alpha_i}\right)^{\rho/(\rho-1)}\right]^{-(\mu(1-\mu)^{-1}(\rho-1))/\rho},$$

and the derived demand functions by[24]

$$X_i = \mu^{(1-\mu)^{-1}}\left[\sum_{i=1}^{m} \alpha_i \left(\frac{q_i}{\alpha_i}\right)^{\rho/(\rho-1)}\right]^{((1-\mu)^{-1}(\mu-\rho))/\rho} \left(\frac{q_i}{\alpha_i}\right)^{1/(\rho-1)}, \qquad i = 1,\ldots,m.$$

[22]There is no constant of integration because by Theorem (II-1), $G(\mathbf{q})$ is homogeneous.
[23]Equivalent expressions have been obtained by McFadden (1966) and Nerlove (1967).
[24]See the discussion in Lau (1969c, pp. 30–33).

However, if some inputs are fixed, then the normalized profit function corresponding to a C.E.S. production function may not have a closed form solution. It is, however, still implicitly defined.

4.4. *Combination C.E.S. – Cobb–Douglas Production Function*

In view of the analytic intractability of the C.E.S. production function with fixed input levels, functions which are hybrids of the C.E.S. and the Cobb–Douglas functions may be used. Some examples are[25]

(1) $$Y = \prod_{i=1}^{m} X_i^{\alpha_i} \left[\sum_{j=1}^{n} \beta_j Z_j\right]^{\mu_2/\rho},$$

with

$$\Pi^* = (1-\mu_1) \prod_{i=1}^{m} \left(\frac{q_i}{\alpha_i}\right)^{-\alpha_i(1-\mu_1)^{-1}} \left[\sum_{j=1}^{n} \beta_j Z_j^{\rho}\right]^{(\mu_2(1-\mu_1)^{-1})/\rho},$$

where

$$\mu_1 = \sum_{i=1}^{m} \alpha_i < 1;$$

(2) $$Y = \left[\sum_{i=1}^{m} \alpha_i X_i^{\rho}\right]^{\mu/\rho} \prod_{j=1}^{n} Z_j^{\beta_j},$$

with

$$\Pi^* = \mu^{\mu/(1-\mu)}(1-\mu)\left[\sum_{i=1}^{m} \alpha_i \left(\frac{q_i}{\alpha_i}\right)^{\rho/(\rho-1)}\right]^{-\mu/(1-\mu)\cdot(\rho-1)/\rho} \left[\prod_{j=1}^{m} Z_j^{\beta_j}\right]^{(1-\mu)^-}$$

(3) $$Y = \left[\sum_{i=1}^{m} \alpha_i X_i^{\rho_1}\right]^{\mu_1/\rho_1} \left[\sum_{j=1}^{n} \beta_j Z_j^{\rho_2}\right]^{\mu_2/\rho_2},$$

with

$$\Pi^* = \mu_1^{\mu_1(1-\mu_1)}(1-\mu_1)\left[\sum_{i=1}^{m} \alpha_i \left(\frac{q_i}{\alpha_i}\right)^{\rho_1/(\rho_1-1)}\right]^{-\mu_1(1-\mu_1)\cdot(\rho_1-1)\rho_1} \times \left[\sum_{j=1}^{n} \beta_j Z_j^{\rho_2}\right]^{(\mu_2/\rho_2)\cdot 1/(1-\mu_1)}.$$

Alternatively, one may specify the normalized restricted profit function directly.

[25]The dual functions may be derived by using either Theorem II-3 or the composition theorems given in Chapter I.1.

4.5. *Quadratic Production Function*

Thus far we have considered only those production functions which satisfy our assumptions globally. However, we shall now consider some production functions (and normalized restricted profit functions) which satisfy our assumptions only over proper convex subsets of R^m (or $R^m \times R^n$ as the case may be). First we consider the quadratic function. Adhering now to the conventions of the multiple-output, multiple-input case, we define

$$L = \alpha_0 + \sum_{i=1}^{m} \alpha_i X_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_{ij} X_i X_j,$$

where X_i is a net output which may be either positive or negative. If the matrix $\mathbf{B} \equiv [\beta_{ij}]$ is required to be positive definite, then L is strongly convex on R^m and in particular on any convex subset of R^m. Monotonicity requires that

$$\boldsymbol{\alpha} + \mathbf{BX} \geqq 0.$$

This system of linear inequalities defines the convex set of $\mathbf{X}$ such that L is monotonic and convex. The constant α_0 may then be adjusted so that L is non-negative on this convex set. Alternatively, one may set $\alpha_0 = 0$ and L is then non-negative, monotonic, and strongly convex on the convex set such that

$$\boldsymbol{\alpha}'\mathbf{X} + \frac{\mathbf{X}'B\mathbf{X}}{2} \geqq 0,$$

$$\boldsymbol{\alpha} + \mathbf{BX} \geqq 0.$$

The quadratic production function has the very convenient property of being *self-dual*, that is, its convex conjugate, the normalized profit function, is also a quadratic function,

$$\Pi^* = -\alpha_0 + \tfrac{1}{2}(\mathbf{p} - \boldsymbol{\alpha})'\mathbf{B}^{-1}(\mathbf{p} - \boldsymbol{\alpha}),$$

where p_i may be the price of a net output or net input. The domain of Π^* is given by the support function of the domain of L, defined above.

The derived supply and demand functions for all commodities other than L are linear functions

$$\mathbf{X} = \frac{\partial G}{\partial \mathbf{p}}$$
$$= \mathbf{B}^{-1}(\mathbf{p} - \boldsymbol{\alpha}),$$

and

$$L = \alpha_0 - \tfrac{1}{2}(\mathbf{p}-\boldsymbol{\alpha})'\mathbf{B}^{-1}(\mathbf{p}-\boldsymbol{\alpha}) + \mathbf{p}'\mathbf{B}^{-1}(\mathbf{p}-\boldsymbol{\alpha})$$
$$= \alpha_0 + \tfrac{1}{2}(\mathbf{p}-\boldsymbol{\alpha})'\mathbf{B}^{-1}(\mathbf{p}-\boldsymbol{\alpha}) + \boldsymbol{\alpha}'\mathbf{B}^{-1}(\mathbf{p}-\boldsymbol{\alpha}).$$

The quadratic function may be further generalized as follows. Suppose that the production function is given by

$$L = \frac{1}{\theta}(\mathbf{X}'\mathbf{B}\mathbf{X})^{\theta/2}, \qquad 1 < \theta < +\infty.$$

Then the normalized profit function is given by

$$\Pi^* = \frac{1}{\eta}(\mathbf{p}'\mathbf{B}^{-1}\mathbf{p})^{\eta/2}, \qquad 1 < \eta < +\infty,$$

where

$$\frac{1}{\theta} + \frac{1}{\eta} = 1.$$

Again, the domain of L and its conjugate Π^* need to be appropriately restricted. If the monotonicity assumption is maintained then $\mathbf{BX} \geqq 0$ defines the domain of L.

4.6. *The Exponential Production Function*

Let

$$Y = 1 - e^{-X}, \qquad X \geqq 0.$$

Then

$$\Pi^* = 1 - q + q \ln q,$$
$$X = -\ln q.$$

We note that for $q > 1$, there is no solution X such that $X \geqq 0$.

4.7. *The "Addilog" Normalized Profit Function*

We next consider normalized profit functions for which an explicit dual production function does not exist. One such example is the indirect addilog function introduced by Houthakker (1960). The normalized profit function is given by

$$\Pi^* = \sum_{i=1}^{m} \alpha_i q_i^{-\beta_i}.$$

The derived supply and demand functions are given by

$$Y = \sum_{i=1}^{m} \alpha_i q_i^{-\beta_i} + \sum_{i=1}^{m} \alpha_i \beta_i q_i^{-\beta_i}$$

$$= \sum_{i=1}^{m} \alpha_i (1 + \beta_i) q_i^{-\beta_i},$$

$$X_i = \alpha_i \beta_i q_i^{-(\beta_i + 1)}, \qquad i = 1,\dots,m.$$

The restrictions for monotonicity and convexity for the single-output case are

$$\alpha_i \beta_i > 0, \qquad \beta_i > -1.$$

4.8. *Reciprocal Quadratic Normalized Profit Function*

$$\Pi^* = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_{ij} q_i^{-1} q_j^{-1}.$$

The derived supply and demand functions have a remarkably simple form

$$Y = \frac{3}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_{ij} q_i^{-1} q_j^{-1},$$

$$X_i = q_i^{-2} \sum_{j=1}^{m} \beta_{ij} q_j^{-1}, \qquad i = 1,\dots,m.$$

It may be verified directly that if $\beta_{ij} \geqq 0$, $\forall i,j$, then Π^* is non-negative, non-increasing, and convex. Also, by Theorem II-1, the production function must be homogeneous of degree $\frac{2}{3}$.

A generalization of this normalized profit function exists with the exponent of q_i equal to $-\mu$, $\mu > 0$. In this latter case, the production function must be homogeneous of degree $2\mu/(1 + 2\mu)$.

4.9. *Transcendental Logarithmic Normalized Production Function*

For the sake of completeness, one should also mention the transcendental logarithmic normalized profit function introduced by Christensen, Jorgenson and Lau (1971 and 1973). The normalized profit function is given by

$$\ln \Pi^* = \alpha_0 + \sum_{i=1}^{m} \alpha_i \ln q_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_{ij} \ln q_i \ln q_j.$$

The demand functions are given by

$$\frac{q_i X_i}{\Pi^*} = -\left(\alpha_i + \sum_{j=1}^{m} \beta_{ij} \ln q_j\right), \qquad i = 1,\ldots,m.$$

Note the remarkably simple estimating form. This function is not a globally valid normalized profit function as it may become non-monotonic or non-convex at some prices. However, it is possible to verify whether the function is monotonic and convex over some convex set of normalized prices. In addition, it has the advantage that it provides a second order approximation to an arbitrary normalized profit function (and hence to an arbitrary concave technology). It can also attain any value of the elasticity of substitution between any pair of inputs.

5. Applications of the Normalized Profit Function

5.1. *Elasticities of Substitution*

As is well-known, many different elasticities of substitution may be defined in the case of a technology which involves more than two inputs,[26] depending on which variables are held constant. A natural definition, however, in the spirit of the Allen–Uzawa definition of the elasticities of substitution for the case of three or more inputs, is the following:

$$\sigma_{ij} = \frac{GG_{ij}}{G_i G_j},$$

which, by the dual transformation, is equivalent to

$$\sigma_{ij} = \frac{-\left(F - \sum_{i=1}^{m} (\partial F / \partial X_i) X_i\right)}{X_i X_j} \frac{F_{ij}^{+}}{|F|},$$

where F_{ij}^{+} is the j,ith cofactor of the matrix F since $[G_{ij}] = -[F_{ij}]^{-1}$, and $|F|$ is the determinant of the Hessian matrix of F.

[26]See, for instance, McFadden (1963), Nerlove (1967) and Uzawa (1962).

In any event, when there are more than two inputs, the elasticities of substitution are not necessarily the most convenient measures of substitutibility.[27] An alternative is provided by own and cross-price elasticities of demand. In what follows, we give characterization theorems for own and cross-price elasticities of demand by solving systems of partial differential equations for the normalized profit function.

Theorem V-1. A production function is Cobb–Douglas if and only if all the own and cross-price elasticities of factor demands are constants.

Proof: Suppose

$$\frac{\partial \ln X_i}{\partial \ln q_j} = k_{ij}, \quad \text{a constant}, \quad \forall i,j.$$

Integrating this system of partial differential equations, we obtain

$$\ln X_i = \sum_l k_{il} \ln q_l + k_{i0}, \qquad i = 1,\ldots,m$$

or

$$X_i = -\frac{\partial G}{\partial q_i} = \mathrm{e}^{k_{i0}} \prod_l q_l^{k_{il}}, \qquad i = 1,\ldots,m,$$

which upon integration yields the Cobb–Douglas normalized profit function. The k_{ij} constants of the different equations may be shown to be the same by making use of the fact that

$$\frac{\partial X_i}{\partial q_j} = -\frac{\partial^2 G}{\partial q_j \partial q_i} = \frac{\partial X_j}{\partial q_i}, \qquad i \neq j.$$

The converse is obvious. Q.E.D.

Theorem V-2. A production function is homogeneous of degree k up to an additive constant if and only if the sum of own and cross-price elasticities of demand is constant for any one commodity.

Proof: Homogeneity of the production function up to an additive constant implies homogeneity of the normalized profit function up to an

[27]Besides, they are insufficient as a description of the technology. See Lau (1976b).

additive constant by Theorems II-1 and II-4. Hence, derived demand is also homogeneous. Hence,

$$\sum_{j=1}^{m} \frac{\partial X_i}{\partial q_j} \frac{q_j}{X_i} = \sum_{j=1}^{m} G_{ij} \frac{q_j}{G_i} = k, \quad \forall i.$$

The converse is proved similarly by retracing the steps and using Euler's Theorem. If G_i is homogeneous then G must be a homogeneous function plus a constant. Q.E.D.

Theorem V-3. A production function is of the Leontief type, that is, the normalized profit function has the form

$$\Pi^* = g\left(\sum_{i=1}^{m} \alpha_i q_i\right),$$

if and only if

$$\frac{\partial \ln X_i}{\partial \ln q_k} = \phi_k(\mathbf{q}), \qquad \begin{array}{l} i = 1,\ldots,m, \\ k = 1,\ldots,m. \end{array}$$

In other words, this means that the elasticities of each of the demand functions with respect to the kth normalized price are identical.

Proof: Necessity is obvious. To prove sufficiency, we first integrate

$$\frac{\partial \ln X_i}{\partial \ln q_k} = \phi_k(\mathbf{q}),$$

to obtain

$$\ln X_i = \int \phi_k(\mathbf{q}) d \ln q_k + \Phi_i^k(\mathbf{q}_{-k}), \qquad \forall i,$$

where $\mathbf{q}_{-k}$ is $\mathbf{q}$ reduced by q_k.

Now

$$\ln X_i - \ln X_k = \Phi_i^k(\mathbf{q}_{-k}) - \Phi_k^k(\mathbf{q}_{-k}),$$

and

$$\ln X_k - \ln X_i = \Phi_k^i(\mathbf{q}_{-i}) - \Phi_i^i(\mathbf{q}_{-i}).$$

Moreover

$$\ln X_i - \ln X_k = \Phi_i^j(\mathbf{q}_{-j}) - \Phi_k^j(\mathbf{q}_{-j}).$$

But the left-hand side is the same expression, aside from a sign change, and we have seen the right-hand side being independent of q_i, q_j and q_k. We conclude that it must be constant. Thus, $\ln (X_i/X_k) = \theta_i/\theta_k$, a constant

or

$$\frac{1}{\theta_i^*}\frac{\partial G}{\partial q_i} = \frac{1}{\theta_k^*}\frac{\partial G}{\partial q_k}, \qquad \forall i,k,$$

with the general solution

$$G(\mathbf{q}) = g\left(\sum_{i=1}^{m} \theta_i^* q_i\right). \qquad \text{Q.E.D.}$$

Corollary 3.1. If in addition $\sum_{k=1}^{m} \phi_k(q)$ is constant, then Π^* is homogeneous up to an additive constant and G has the form $G = [\sum_{i=1}^{m} \alpha_i q_i]^\mu + \alpha_0$.

Proof: This follows from this theorem and Theorem V-2. Q.E.D.

Theorem V-4. A normalized profit function has the form

$$\Pi^* = g\left(\sum_{i=1}^{m} g_i(q_i)\right),$$

if and only if

$$\frac{\partial \ln X_i}{\partial \ln q_k} = \phi_k(\mathbf{q}), \qquad i \neq k; \quad i,k = 1,\ldots,m.$$

Proof: Repeating the argument used in the previous theorem, one has

$$\ln X_i - \ln X_j = \Phi_i^k(\mathbf{q}_{-k}) - \Phi_j^k(\mathbf{q}_{-k}), \qquad \forall i,j,k, \quad i,j \neq k.$$

Thus, for fixed i,j, one concludes that

$$\ln X_i - \ln X_j = \psi_{ij}(q_i,q_j),$$

and for fixed l

$$\ln X_i - \ln X_l = \psi_{il}(q_i,q_l),$$
$$\ln X_j - \ln X_l = \psi_{jl}(q_j,q_l).$$

Combining the last two equations, one has

$$\ln X_i - \ln X_j = \psi_{il}(q_i,q_l) - \psi_{jl}(q_j,q_l),$$

but the right-hand side must be independent of q_l for all values of q_i and q_j.

Hence

$$\phi_{il} = \phi_i(q_i) - \phi_l(q_l), \qquad \forall i,l.$$

Thus from

$$\ln X_i/X_j = \phi_i(q_i) - \phi_j(q_j),$$

we obtain

$$\frac{\partial G/\partial q_i}{\partial G/\partial q_j} = \frac{\phi_i(q_i)}{\phi_j(q_j)},$$

which may be integrated as

$$G(\mathbf{q}) = g\left(\sum_{i=1}^{m} g_i(q_i)\right). \qquad \text{Q.E.D.}$$

Corollary 4.1. If in addition, $\Sigma_{k=1}^{m} (\partial \ln X_i)/(\partial \ln q_k)$ is a constant, then Π^* is homogeneous up to an additive constant, and G has the C.E.S. form

$$G = \left[\sum_{i=1}^{m} \alpha_i q_i^{\rho}\right]^{\mu/\rho} + \alpha_0.$$

Proof: This follows from Theorems V-2 and V-4, and the fact that an additive function is homogeneous if and only if it has the C.E.S. form. Q.E.D.

With these results then, one can examine directly the own and cross-price elasticities of demand, that is, the comparative statics, and obtain an idea of the degree of substitution. The aforementioned results also apply, with appropriate modification, to subsets of the inputs.

5.2. *Technical Change*

Technical change may be represented by a production function

$$Y = F(\mathbf{X},t), \qquad \frac{\partial F}{\partial t} \geqq 0.$$

This gives rise to a normalized profit function

$$\Pi^* = G(\mathbf{q},t), \qquad \frac{\partial G}{\partial t} \geqq 0.$$

By duality, $\partial F/\partial t = \partial G/\partial t$ at the profit maximum. Hence for given $\mathbf{q}$ normalized profit increases with time.

Definition. A production function is *Hicks neutral* if it can be written in the form

$$Y = F(f(\mathbf{X}),t).$$

Definition. A production function is *Harrod neutral* if it can be written in the form

$$Y = F(f(L,t),\mathbf{X}),$$

where L is labor, the primary factor of production.

Definition. A normalized profit function is *indirectly Hicks neutral* if it can be written in the form

$$\Pi^* = G(f(\mathbf{q}),t).$$

Definition. A normalized profit function is *indirectly Harrod neutral* if it can be written in the form

$$\Pi^* = G(f(w,t),\mathbf{q}).$$

The practical implication of Hicksian neutrality is that the ratio of the marginal products of any two inputs is independent of time. The practical implication of indirect Hicksian neutrality is that the ratios of the derived demands of any two inputs is independent of time.

It should be noted that in general direct Hicksian neutrality does not imply indirect Hicksian neutrality or *vice versa*. A technology is both directly and indirectly Hicksian neutral only if either it is homothetic or it is additive in t. This follows immediately from Theorem II-15. Also, under homotheticity, direct Hicksian neutrality implies and is implied by indirect Hicksian neutrality.

The practical implication of Harrod neutrality is that the ratio of the marginal productivity of labor to the rate of technical change measured in terms of output is independent of $\mathbf{X}$. The practical implication of indirect Harrod neutrality is that the ratio of the demand for labor to the rate of technical change measured in terms of normalized profit is independent of **q**. In general, direct Harrod neutrality does not imply indirect Harrod neutrality or *vice versa*. A production function is both directly and indirectly Harrod neutral only under one of the two following conditions:

(1) $f(L,t) = f(A(t)L)$,

or

(2) $Y = F(\mathbf{X}) + f(L,t)$.

That these conditions are sufficient is obvious. That they are necessary may be shown as follows:

Let $Y = F(f(L,t),\mathbf{X})$. Let $\Pi^* = G(\bar{f},\mathbf{q})$ be the normalized restricted profit function corresponding to F with $F(L,t) = \bar{f}$. The normalized profit function with f unrestricted is then given by

$$\Pi^* = \sup_{f}\{G(\bar{f},\mathbf{q}) - wh(\bar{f},t)\},$$

where $h(\bar{f},t)$ is the inverse function of $f(L,t)$ for each given t. The necessary condition for a maximum is

$$\frac{\partial G}{\partial \bar{f}}(\bar{f},\mathbf{q}) - w\frac{\partial h}{\partial \bar{f}}(\bar{f},t) = 0.$$

Differentiating Π^* with respect to t and w we obtain

$$\frac{\partial \Pi^*}{\partial w} = -w\frac{\partial h}{\partial t}, \quad \frac{\partial \Pi^*}{\partial w} = -h(\bar{f},t).$$

Hence

$$\frac{\partial \Pi^*/\partial t}{\partial \Pi^*/\partial w} = w\frac{\partial \ln h}{\partial t},$$

$$\frac{\partial}{\partial q}\left(\frac{\partial \Pi^*/\partial t}{\partial \Pi^*/\partial w}\right) = w\cdot\frac{\partial^2 \ln h}{\partial t\,\partial \bar{f}}\cdot\frac{\partial \bar{f}}{\partial \mathbf{q}} = 0.$$

Thus, either $(\partial^2 \ln h)/(\partial t\,\partial \bar{f}) = 0$ which implies that

$$h = h^*(\bar{f})A^*(t) = L,$$

and hence

$$f = h^{*-1}(A(t)L).$$

Or $\partial \bar{f}/\partial \mathbf{q} = 0$, which implies, by differentiating the first-order condition implicitly, that

$$G(\bar{f},\mathbf{q}) = g(\bar{f}) + G(\mathbf{q}),$$

and hence

$$F(f(L,t),\mathbf{X}) = F(\mathbf{X}) + f(L,t).$$

We note that the first condition (1) corresponds precisely to that of labor-augmenting technical change. One possible specialization of the form of technical change is factor- or output-augmentation. Under factor and output augmenting technical change the production function may be written as

$$Y = A(t)F(A_1(t)X_1,\dots,A_m(t)X_m).$$

Thus $A(t)$ represents "output-augmenting" technical change and $A_i(t)$'s represent "factor-augmenting" technical change. If $A_i(t) = A^*(t), \forall i$, and F is a homothetic function, one can write

$$Y = A(t)F(A^*(t)^{-1}H(X_1,\dots,X_m)),$$

which is clearly Hicksian neutral. It reduces to Harrod neutral technical change if and only if $A(t)$ and $A_i(t)$'s are all constants except for the $A_i(t)$ associated with labor, the primary factor.

With "commodity-augmenting" technical change, the normalized profit function is given by Theorem II-3 as

$$\Pi^* = A(t)G\left(\frac{q_1}{A_1(t)}A(t),\dots,\frac{q_m}{A_m(t)}A(t)\right),$$

where $G(\mathbf{q})$ is the normalized profit function corresponding to $F(X)$. Thus, the production function is "commodity-augmenting" if and only if the normalized profit function is "commodity-augmenting". It is also clear that factor-augmentation has the same effect as price-diminution.

A technical change process is "factor-1 augmenting" if the production function may be written in the form

$$Y = F(A_1(t)X_1,X_2,\dots,X_m).$$

A given technical change process is "factor-1 saving" if

$$\frac{\partial X_1}{\partial t} = \frac{\partial}{\partial t} - \frac{\partial G}{\partial q_1}$$
$$= -G_{1t} < 0.$$

Under what conditions is a technical change process simultaneously "factor-1 augmenting" and "factor-1 saving"? We note that under factor-1 augmentation the normalized profit function can be written as

$$\Pi^* = G(q_1/A_1(t), q_2, \ldots, q_m).$$

Thus

$$\begin{aligned}\frac{\partial X_1}{\partial t} &= \frac{\partial}{\partial t} - \frac{\partial G}{\partial q_1}\\ &= G_{11}\frac{q_1}{A_1^2}\frac{\dot{A}_1}{A_1} + G_1\frac{1}{A_1}\frac{\dot{A}_1}{A_1}.\end{aligned}$$

In order for this to be less than zero, we need

$$G_{11}\frac{q_1}{A_1} + G_1 < 0,$$

which implies by a dual transformation that

$$\frac{\partial X_1}{\partial q_1}\cdot q_1 + X_1 < 0,$$

or a derived demand elasticity of X_1 with respect to own price of greater than -1. Thus in general one cannot identify "factor-augmenting" technical change with "factor-saving" technical change. We note that, even with factor augmenting technical change occurring in only one factor, the derived demands of the other inputs may also change over time as

$$\begin{aligned}\frac{\partial}{\partial t}X_j &= \frac{\partial}{\partial t} - \frac{\partial G}{\partial q_j} \qquad j \neq 1,\\ &= G_{1j}q_1\frac{\dot{A}}{A^2},\\ \frac{\partial Y}{\partial t} &= \frac{\partial}{\partial t}\left(G - \sum_{j=1}^{n} q_j \frac{\partial G}{\partial q_j}\right)\\ &= \left(\sum_{j=1}^{n} G_{1j}q_j\frac{\dot{A}}{A^2}\right).\end{aligned}$$

We note further that if G_1 were homogeneous of degree $-k$ (necessarily so because G_1 must be negative) then $\sum_{j=1}^{n} G_{1j}q_j = -kG_1$, or if G were additive, supply must be increasing. In general $\partial Y/\partial t$ is indeterminate in sign.

5.3. *Relative Efficiency*[28]

There are two dimensions to the problem of efficiency: technical efficiency and price efficiency. A firm is technically more efficient than another firm, if, and only if, it consistently produces a higher output given identical inputs for both firms. A firm is price-efficient, if, and only if, the value of the marginal product of each input is equated to its price. Any departure from this equality implies price inefficiency. It is sometimes desirable to compare the relative degree of technical efficiency and also the relative degree of price-inefficiency across two firms. If a firm is price-efficient, its profit is at a maximum for a given level of technical efficiency. Thus, a natural measure of relative price efficiency is the relative level of actual profits. A firm is considered to be more price-efficient, if, given the same prices of inputs and outputs and the same degree of technical efficiency, it is more profitable than another firm. Based on this definition, the technically more efficient firm which is also price-efficient will always be more profitable than another firm which is only price-efficient. It is important to note that relative technical efficiency need not imply relative price efficiency and *vice versa*.

Straightforward tests of relative technical and price efficiency between two firms (or groups of firms) may be devised on the basis of the normalized profit function. It is clear that given comparable endowments, identical technology, and normalized input prices, the actual normalized profits of the two firms should be identical if they both have maximized profits. To the extent that one is more price efficient, or technically more efficient, than the other, the normalized profits will differ even for the same normalized input prices and endowments of fixed inputs. The actual normalized profit functions will hence be different for the two firms.

Let us represent the situation as follows: For each firm, the marginal conditions are given by

$$\frac{\partial A_1 F(\mathbf{X}_1,\mathbf{Z}_1)}{\partial \mathbf{X}_1} = \mathbf{K}_1\mathbf{q}_1, \qquad \frac{\partial A_2 F(\mathbf{X}_2,\mathbf{Z}_2)}{\partial \mathbf{X}_2} = \mathbf{K}_2\mathbf{q}_2, \tag{V-1}$$

$$\mathbf{k}_1 = \operatorname{diag}[\mathbf{K}_1] \geqq 0, \qquad \mathbf{k}_2 = \operatorname{diag}[\mathbf{K}_2] \geqq 0,$$

where $\mathbf{K}_i$ is a diagonal matrix, $\mathbf{X}_i$, $\mathbf{Z}_i$, $\mathbf{q}_i$ and $\mathbf{k}_i$ are vectors, A_i is a scalar, and the subscript refers to the firm. If both firms are equally efficient in

[28]This section draws heavily on my joint work with P.A. Yotopoulos. See Lau and Yotopoulos (1971), Yotopoulos and Lau (1973), and Yotopoulos, Lau and Lin (1976).

optimizing with respect to all variable inputs, then $\mathbf{k}_1 = \mathbf{k}_2$. If both firms are equally efficient technically, then $A_1 = A_2$. Equation (V-1) reduced to the usual first-order conditions for profit maximization if and only if $\mathbf{k}_1 = \mathbf{k}_2 = [\mathbf{1}]$, a unit vector. Otherwise, they must be interpreted as decision rules for the individual firms. $\mathbf{k}_1$ and $\mathbf{k}_2$ may assume any non-negative values, and in particular, the special values of $[\mathbf{0}]$ and $[\mathbf{1}]$.

That the decision rules for the firm consist of equating the marginal product to a constant times the normalized price of each input may be rationalized as follows: (1) consistent over and under-valuation of the opportunity costs of the resources by the firms; (2) satisficing behavior; (3) divergence of expected and actual normalized prices; (4) divergence of the subjective probability distribution of the normalized prices from the objective distribution of normalized prices; (5) the elements of $\mathbf{k}_i$ may be interpreted as the first-order coefficients of a Taylor's series expansion of arbitrary decision rules of the type

$$\frac{\partial F_i}{\partial X_{ij}} = f_{ij}(q_{ij}),$$

where $f_{ij}(0) = 0$. A wide class of decision rules may be encompassed under (5).

Let $G(\mathbf{q},\mathbf{Z})$ be the normalized profit function corresponding to $F(\mathbf{X},\mathbf{Z})$. The firms then may be regarded to behave as if they maximize normalized profit subject to price vectors $\mathbf{K}_1\mathbf{q}_1/A_1$ and $\mathbf{K}_2\mathbf{q}_2/A_2$, respectively. Their behavior thus may be represented by the "behavioral" normalized profit functions

$$\Pi_1^b = A_1 G(k_{11}q_1/A_1,\ldots,k_{1m}q_m/A_1;Z_{11},\ldots,Z_{1n}),$$

and

$$\Pi_2^b = A_2 G(k_{21}q_1/A_2,\ldots,k_{2m}q_m/A_2;Z_{21},\ldots,Z_{2n}).$$

A test of equal relative efficiency implies a test of the hypothesis $A_1 = A_2$ *and* $\mathbf{k}_1 = \mathbf{k}_2$. The derived demand functions are given by

$$X_{ij} = -A_i \frac{\partial G(\mathbf{K}_i\mathbf{q}/A_i;\mathbf{Z}_i)}{\partial k_{ij}q_j}, \qquad i = 1,2, \quad j = 1,\ldots,m,$$

and the supply functions by

$$Y_i = A_i\left\{G(\mathbf{K}_i\mathbf{q}/A_i;\mathbf{Z}_i) - \sum_{j=1}^{m} k_{ij}q_j\left[\frac{\partial G(\mathbf{K}_i\mathbf{q}/A_i;\mathbf{Z}_i)}{\partial k_{ij}q_j}\right]\right\}.$$

The actual normalized profit functions are given by

$$\begin{aligned} \Pi_i^a &= Y_i - \sum_{j=1}^{m} q_j X_{ij} \\ &= A_i \left\{ G(\mathbf{K}_i \mathbf{q}/A_i;\mathbf{Z}_i) + \sum_{j=1}^{m} (1-k_{ij}) q_j \left[\frac{\partial G(\mathbf{K}_i\mathbf{q}/A_i;\mathbf{Z}_i)}{\partial k_{ij} q_j} \right] \right\}, \quad i = 1,2. \end{aligned}$$

Observe (1) $\partial \Pi^a / \partial A \gtreqless 0$; (2) when $\mathbf{k}_i = [\mathbf{1}]$, the actual and "behavioral" normalized profit functions coincide; and (3) if and only if $A_1 = A_2$ and $k_1 = k_2$, the actual as well as the behavioral normalized profit functions and supply and demand functions of the two firms coincide with each other. This last result is the basis of the null hypothesis for no difference in relative efficiency. When appropriate functional forms are specified for G, the joint hypothesis that $A_1 = A_2$ *and* $\mathbf{k}_1 = \mathbf{k}_2$ may be tested by comparing the coefficient estimates from either the actual profit function or the supply and demand functions, or both.

An additional test becomes relevant if we reject the joint hypothesis that $(A_1,\mathbf{k}_1) = (A_2,\mathbf{k}_2)$. In this case an overall indication of the relative efficiency between the two firms within a specified range of normalized prices for variable inputs may be obtained by comparing the actual values of the normalized profit functions within this range. If

$$\Pi_1^a \geqq \Pi_2^a,$$

for all normalized prices within a specified range, then clearly, the first firm is relatively more efficient within the price range. If some knowledge on the probability distribution of the future prices is available, a choice may be made as to the relative efficiency of the two firms.

One can also test the hypothesis that the fixed inputs command equal rent on the two firms by computing the first derivatives of the actual normalized profit functions with respect to the fixed inputs and testing for their equality. This may have important implications on the optimal form of organization.

Finally the above analysis can be easily extended to three or more firms (or groups of firms). We conclude this subsection with an example.

Example

The normalized restricted profit function corresponding to a Cobb–Douglas production function with m variable inputs and n fixed inputs is, from Section 4.2,

$$\Pi^* = (1-\mu)\left[\prod_{i=1}^{m}\left(\frac{q_i}{\alpha_i}\right)^{-\alpha_i(1-\mu)^{-1}}\right]\left[\prod_{j=1}^{n} Z_j^{\beta_j(1-\mu)^{-1}}\right],$$

$$\mu = \sum_{i=1}^{m} \alpha_i < 1.$$

By direct computation, the actual normalized profit functions and the demand functions are

$$\Pi_i^a = A_i^{(1-\mu)^{-1}}\left(1-\sum_{j=1}^{m}\frac{\alpha_j}{k_{ij}}\right)\left[\prod_{j=1}^{m} k_{ij}^{-\alpha_j(1-\mu)^{-1}}\right]\left[\prod_{j=1}^{m} \alpha_j^{\alpha_j(1-\mu)^{-1}}\right]$$
$$\times\left[\prod_{j=1}^{m} q_j^{-\alpha_j(1-\mu)^{-1}}\right]\left[\prod_{j=1}^{n} Z_j^{\beta_j(1-\mu)^{-1}}\right], \qquad i = 1,2,$$

$$X_{ij} = A_i^{(1-\mu)^{-1}}\left(\frac{\alpha_j}{k_{ij}q_j}\right)\left[\prod_{j=1}^{m} k_{ij}^{-\alpha_j(1-\mu)^{-1}}\right]\left[\prod_{j=1}^{m} \alpha_j^{\alpha_j(1-\mu)^{-1}}\right]$$
$$\times\left[\prod_{j=1}^{m} q_j^{-\alpha_j(1-\mu)^{-1}}\right]\left[\prod_{j=1}^{n} Z_j^{\beta_j(1-\mu)^{-1}}\right], \qquad i = 1,2, \quad j = 1,\ldots,m.$$

From these two equations, one may derive

$$\frac{q_j X_{ij}}{\Pi_i^a} = \frac{\alpha_j / k_{ij}}{\left(1-\sum_{j=1}^{m}\frac{\alpha_j}{k_{ij}}\right)}, \qquad i = 1,2, \quad j = 1,\ldots,m.$$

These actual profit share equations may be combined with the natural logarithm of the actual normalized profit functions to obtain estimates of A_i, $\mathbf{k}_i$ and the technological parameters.

5.4. *Monopolistic Profit Functions*

A monopolist faces a downward sloping demand curve. Let the inverse demand function be given as

$$p = D(Y),$$

where

$$D'(Y) < 0.$$

Then the profit maximization problem becomes

$$\max P = D(F(\mathbf{X},\mathbf{Z}))F(\mathbf{X},\mathbf{Z}) - \mathbf{q}^{*\prime}\mathbf{X},$$

where q^* is the vector of nominal prices of the variable inputs.

Let $S(\mathbf{X},\mathbf{Z}) \equiv D(F(\mathbf{X},\mathbf{Z}))F(\mathbf{X},\mathbf{Z})$ be the revenue function.[29] If it is assumed that $S(\mathbf{X},\mathbf{Z})$ satisfies Assumptions (F.1) through (F.7), then the profit maximization problem is isomorphic to the normalized profit maximization problem in Section 1. Thus, the profit function $G(\mathbf{q}^*,\mathbf{Z})$, is given by

$$G(\mathbf{q}^*,\mathbf{Z}) = \max_{\mathbf{X}} \{S(\mathbf{X},\mathbf{Z}) - (\mathbf{q}^{*\prime}\mathbf{X})\},$$

which satisfies Assumptions (G.1) through (G.7). Moreover, there is a one-to-one correspondence between $S(\mathbf{X},\mathbf{Z})$ and $G(\mathbf{q}^*,\mathbf{Z})$.

All the dual relationships which hold between F and G hold between S and G. As before, the demand functions for the variable inputs are given by

$$\mathbf{X}^* = -\frac{\partial G}{\partial \mathbf{q}^*}(\mathbf{q}^*,\mathbf{Z}),$$

and the optimal revenue function is given by

$$S^* = G(\mathbf{q}^*,\mathbf{Z}) - \mathbf{q}^{*\prime}\frac{\partial G}{\partial \mathbf{q}^*}(\mathbf{q}^*,\mathbf{Z}).$$

We note that G depends on nominal prices of the inputs only and hence X^* and S^* depend on only the nominal prices of the inputs.

For the purpose of econometric applications, one may just as well start with a function $G(\mathbf{q}^*,\mathbf{Z})$ which satisfies Assumptions (G.1) through (G.7) without worrying about the properties of $S(\mathbf{X},\mathbf{Z})$ since as McFadden has emphasized in Chapter I.1, one cannot in fact observe those input vectors for which $S(\mathbf{X},\mathbf{Z})$ fails to satisfy the Assumptions (F.1) through (F.7).

Example

Let

$$p = Y^{-\epsilon}, \qquad 1 > \epsilon > 0,$$

$$Y = \prod_{i=1}^{m} X_i^{\alpha_i}, \qquad \sum_{i=1}^{m} \alpha_i = \mu < 1.$$

Then

[29]Note that the revenue function as used here is different from the revenue function $R(p,\mathbf{Z})$ which gives the *maximized* value of revenue for given p and $\mathbf{Z}$.

$$S = \prod_{i=1}^{m} X_i^{\alpha_i(1-\epsilon)}.$$

The monopolistic profit function is hence

$$\Pi = (1-\mu^*) \prod_{i=1}^{m} \left(\frac{q_i^*}{\alpha_i^*}\right)^{-\alpha_i^*(1-\mu^*)^{-1}},$$

where

$$\alpha_i^* \equiv \alpha_i(1-\epsilon), \qquad i = 1,\dots,m,$$

$$\mu^* \equiv \sum_{i=1}^{m} \alpha_i^* \equiv (1-\epsilon)\mu.$$

The derived demand functions are given by

$$X_i = \alpha_i^* q_i^{*-1} \prod_{j=1}^{m} \left(\frac{q_j^*}{\alpha_j^*}\right)^{-\alpha_j^*(1-\mu^*)^{-1}}, \qquad i = 1,\dots,m.$$

Revenue is given by

$$R = S = \prod_{i=1}^{m} \left(\frac{q_i^*}{\alpha_i^*}\right)^{-\alpha_i^*(1-\mu^*)^{-1}}.$$

Finally, we note that while given the profit function alone one can construct an $S(\mathbf{X},\mathbf{Z})$ through the conjugacy operation, one cannot identify $F(\mathbf{X},\mathbf{Z})$ without additional information. We should also emphasize that the assumption of concavity of $S(\mathbf{X},\mathbf{Z})$ neither implies nor is implied by the concavity of $F(\mathbf{X},\mathbf{Z})$. In fact, if there is indeed a monopoly, it is likely that $F(\mathbf{X},\mathbf{Z})$ is non-concave in $\mathbf{X}$.

5.5. *Dynamic Behavior*

Dynamic models have been introduced into econometric research via two principal hypotheses – the "adaptive expectations" hypothesis and the "lagged adjustment" hypothesis. These hypotheses can be readily incorporated into the normalized profit function approach.

5.5.1. *Adaptive Expectations Hypothesis*

The firm is assumed to maximize profit for given expected normalized prices. Then, for a given technology, there is a normalized profit function of expected normalized prices which are in turn functions of current and past normalized prices. Let the price expectation formation process be

$$q_i^0(t) = \omega_i(L)q_i(t), \qquad i = 1,\dots,m,$$

where $q_i^0(t)$ is the expected normalized price of the ith input, and $\omega_i(L)$ the rational distributed lag operator for the ith price.[30]

The expected normalized profit function is then given by

$$\Pi^{*0} = G(q_1^0,\dots,q_m^0).$$

Supply and demand as functions of expected normalized prices are given by

$$Y^0 = G(q_1^0,\dots,q_m^0) - \sum_{i=1}^{m} \frac{\partial G}{\partial q_i^0} q_i^0,$$

$$X_i^0 = -\frac{\partial G}{\partial q_i^0}, \qquad i = 1,\dots,m.$$

In general, both Y^0 and X_i^0's are functions of both current and past prices, with the time structure of the effect of different input prices given by the coefficients of $\omega_i(L)$. Actual normalized profit, on the other hand, is given by

$$P^* = Y^0 - \sum_{i=1}^{m} \frac{\partial G}{\partial q_i^0} q_i.$$

5.5.2. *Lagged Adjustment Hypothesis*

Lagged adjustment models are in general based on an adjustment equation,

$$X_t - X_{t-1} = \omega(L)[X_t^* - X_{t-1}], \tag{V-2}$$

where X_t^* is the desired quantity in period t and the subscripts i are suppressed. Equation (V-2) may be rewritten as

$$(1-L)X_t = \omega(L)X_t^* - \omega(L)LX_t,$$

$$(1-L+\omega(L)L)X_t = \omega(L)X_t^*,$$

$$X_t = \frac{\omega(L)}{(1-L+\omega(L)L)} X_t^*$$

$$= \mu(L)X_t^*,$$

[30]For an exposition of rational distributed lag functions and rational distributed lag operators, see Jorgenson (1966a).

where

$$\mu(L) \equiv \frac{\omega(L)}{(1 - L + \omega(L)L)}.$$

Let G be the normalized profit function for the technology in period t; then

$$Y^* = G - \sum_{i=1}^{m} \frac{\partial G}{\partial q_i} q_i,$$

$$X_i^* = -\frac{\partial G}{\partial q_i}, \qquad i = 1,...,m.$$

However, the actual supply and demand equations are given by

$$X_i = -\mu_i(L)\frac{\partial G}{\partial q_i}, \qquad i = 1,...,m,$$

and

$$Y = F\left(\mu_1(L)\frac{\partial G}{\partial q_1},...,\mu_m(L)\frac{\partial G}{\partial q_m}\right).$$

Both the "adaptive expectations" and the "lagged adjustment" models represent attempts to introduce dynamic elements into a basically static concept and are not completely satisfactory. There is, in principle, no reason why truly dynamic "profit" functions cannot be constructed. These will be functions which give the maximized value of the net worth (or equivalently the present value) of the firm for specified current and future expected prices and initial endowments.

The net worth function, or functional, may be written as

$$NW = G(\mathbf{p},\mathbf{q},t),$$

where $\mathbf{p}$ and $\mathbf{q}$ are possibly infinite dimensional vectors. The profit in period t_i is given by

$$\Pi^* = G(\mathbf{p},\mathbf{q},t_i) - G(\mathbf{p},\mathbf{q},t_i - 1).$$

The supply and demand functions in period t_i may be obtained by the usual duality relationships.

Before such a dynamic "profit" function can be constructed, however, one must have a well-developed theory of intertemporal production. The dual to the dynamic profit function is the production function that links output and input possibilities of all periods, with due recognition given to the fact that future inputs cannot contribute to present output.

Given a dynamic "profit" function, the complete optimal production and investment plan for the future may be calculated based on the expectations of future price movements. These dynamic profit functions must satisfy certain structural characteristics, e.g., at each point t in time different from the point of planning, the profit function must be expressible as a function of endowments at time t and the price at time t and in the future. The supply and demand functions at time t will be expressible as functions independent of the past prices. One may also want to impose the requirement of stationarity, a concept introduced by Koopmans et al. (1964), with regard to dynamic profit functions.

In Chapter II.4 of this volume Fuss and McFadden also analyze the problem of intertemporal production using duality concepts.

5.6. *Profit Functions and Uncertainty*

Using the normalized profit function, one can obtain an immediate proof of a well-known result that randomness in prices results in higher expected profits if the firm is able to adjust instantaneously than if the prices are constant and equal to their expected values.[31] Expected normalized profits are given by

$$E[G(\mathbf{q})].$$

Normalized profits at expected normalized prices are given by

$$G(E[\mathbf{q}]).$$

By Jensen's (1906) inequality on convex functions, one obtains immediately that

$$E[G(\mathbf{q})] \geqq G(E[\mathbf{q}]).$$

Note that this result holds true for fluctuations in all prices and not only in the output prices as the problem is customarily posed.

The effect of randomness on expected output, on the other hand, is not clear cut,

$$Y = G - \sum_{i=1}^{m} \frac{\partial G}{\partial q_i} q_i,$$

$$\begin{aligned} E[Y] &= E[G(\mathbf{q})] - \sum_{i=1}^{m} E\left[\frac{\partial G}{\partial q_i} q_i\right] \\ &\gtrless G(E[\mathbf{q}]) - \sum_{i=1}^{m} E[q_i] \frac{\partial G}{\partial q_i}(E[\mathbf{q}]) \\ &= Y(E[\mathbf{q}]). \end{aligned}$$

[31]See, for example, Oi (1961).

However, if it is assumed that the production function is homogeneous of degree k, then by Theorem II-1,

$$Y = (1-k)^{-1}G.$$

Hence

$$\begin{aligned} E[Y] &= (1-k)^{-1}E[G(\mathbf{q})] \\ &\geqq (1-k)^{-1}G(E[\mathbf{q}]) \\ &\geqq Y(E[\mathbf{q}]). \end{aligned}$$

Expected output is also increased by randomness in both output and input prices.

6. Summary and Conclusions

In the preceding sections, the potential usefulness of the concept of the normalized profit function in both theoretical and empirical applications has been demonstrated. In particular, the normalized profit function provides a convenient and logical link, by virtue of its duality properties, between theoretical specification of a model and empirical implementation. By deriving a system of supply and demand functions from a normalized profit function, rather than attempting to solve the profit maximization problem itself, one avoids the potential difficulties (sometimes impossibility) of obtaining closed form solutions. Nevertheless, one is assured that the supply and demand functions thus derived do correspond to those that are obtained through the maximization of profits subject to some production function with the usual regularity properties. Many additional factors, such as imperfection of markets and technical change, may also be conveniently introduced in a straightforward way. Alternatively, given an arbitrary system of supply and demand functions, one can verify their consistency with profit maximization subject to a production function constraint by checking whether the system is integrable into a normalized profit function.

In addition, it should be emphasized that the normalized profit function contains *all* the empirically relevant information. Supply and demand functions derived from a normalized profit function satisfy all the *a priori* restrictions imposed by the production function. Hence there is no loss in generality, but a gain in elegance and analytical convenience, if one starts out with a normalized profit function.

Finally, through the examples provided, it may be seen that a large number of complete systems that (1) approximate any arbitrary normalized profit (and hence production) function, (2) can attain any value of elasticity of substitution between any pairs of commodities, and (3) are econometrically convenient to estimate – meaning in most cases linear in parameters – are available. They offer greater flexibility than the supply and demand systems traditionally used in the literature. This greater flexibility may result in more realistic modeling of the economy or the firm by making indispensable restrictive assumptions introduced for the sake of obtaining closed form solutions.

The potentials for profit function (and revenue and cost functions) are by no means exhausted here. Directions for future research include (1) dynamic models, (2) incorporation of adjustment costs, (3) non-classical technologies, (4) profit maximization under uncertainty, and (5) departures from profit-maximizing behavior.