Econ 240A: Problem Set 5
Solutions to Selected Problems from Chapter 4

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1.

Note that $S \sim \text{Unif}(0,1)$. Assume the sequences of random variables with components $X_n = S^n$, $Y_n = nS^n$, $Z_n = \log X_n$, are all iid.

We have

$$\begin{aligned} E[X^2] &= E[S^{2n}] \\ &= \int_0^1 s^{2n} ds \\ &= \frac{1}{2n+1} \\ &\to 0, \end{aligned}$$

so $X_n \stackrel{\mathcal{L}^2}{\to} 0$, which implies that $X_n \stackrel{p}{\to} 0, X_n \stackrel{d}{\to} 0$.

Since X_n converges to zero in probability, its almost sure limit must also be zero if such a limit can be shown to exist. Do we in fact have $X_n \stackrel{as}{\to} 0$? Note that $X_n \stackrel{as}{\to} 0$ iff $\forall \epsilon > 0, P(|X_k| < \epsilon : k = n+1, n+2, \ldots) \to 1$. If $\epsilon \in (0,1)$ we have

$$\begin{aligned} P(|X_k| < \epsilon : k = n+1, n+2, \ldots) &= \prod_{k=n+1}^{\infty} P(S < \epsilon^{\frac{1}{k}}) \\ &= \prod_{k=n+1}^{\infty} \epsilon^{\frac{1}{k}} \\ &= \epsilon^{\sum_{k=n+1}^{\infty} \frac{1}{k}} \\ &= 0, \end{aligned}$$

so X_n does not converge almost surely to 0.

We can show that Y_n converges in probability to zero. For any $\epsilon > 0$ we have

$$\begin{aligned} P(|Y_n| \geq \epsilon) &= 1 - P\left(S < \left(\frac{\epsilon}{n}\right)^{\frac{1}{n}}\right) \\ &= \begin{cases} 1 - \left(\frac{\epsilon}{n}\right)^{\frac{1}{n}} \to 0, & \epsilon < n \\ 1 - 1 = 0, & \epsilon \geq n \end{cases} \end{aligned}$$

This implies that $Y_n \xrightarrow{d} 0$ as well.

Next, we check for the almost sure or $\mathcal{L}^2$-convergence of Y_n to zero. We have

$$\begin{aligned} E[Y_n^2] &= E[n^2 S^{2n}] \\ &= \int_0^n n^2 s^{2n} ds \\ &= \frac{n^2}{2n+1} \\ &\to \infty, \end{aligned}$$

so Y_n does not converge in quadratic mean. Now pick $\epsilon \in (0,1)$. We have

$$\begin{aligned} P(|Y_k| < \epsilon : k = n+1, n+2, \ldots) &= \prod_{k=n+1}^{\infty} P\left(S < \left(\frac{\epsilon}{k}\right)^{\frac{1}{k}}\right) \\ &= \epsilon^{\sum_{k=n+1}^{\infty}} \prod_{k=n+1}^{\infty} \left(\frac{1}{k}\right)^{\frac{1}{k}} \\ &\to 0 \cdot 1 = 0, \end{aligned}$$

which shows that Y_n does not converge almost surely to zero.

Finally, we can show that $Z_n = \log S^n$ does not have a stochastic limit by showing that it fails to converge in distribution. For $z \in (-\infty, 0]$ we have

$$\begin{aligned} P(Z_n \leq z) &= P(S \leq e^{\frac{z}{n}}) \\ &= e^{\frac{z}{n}} \\ &\to 1, \end{aligned}$$

so the cdf of Z_n converges to a function that assigns the value of 1 to every point in the support of Z_n. Clearly this function is not a cdf.[1]

[1]Note that if the support of Z_n were bounded, this type of limiting behavior for the cdf would be compatible with convergence in probability (and distribution) to the infimum of the support, as is the case with the cdfs of X_n and Y_n, which have supports $[0,1]$ and $[0,n]$ respectively.

2.

Note that S_n is normal with mean zero and variance $V_n = \sigma^2[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}]$. This implies that the sample mean $\frac{S_n}{n} \sim N(0, W_n)$, where $W_n = \frac{1}{n^2}V_n \to \infty$. We can prove that $\frac{S_n}{n}$ does not converge in probability by showing that it does not converge in distribution. Note that $\frac{\frac{S_n}{n}}{\sqrt{W_n}} \sim N(0,1)$. Let $\Phi(\cdot)$ be the cdf of a $N(0,1)$ distribution. We have for any $s \in \Re$,

$$\begin{aligned} P(\frac{S_n}{n} \leq s) &= P(\frac{\frac{S_n}{n}}{\sqrt{W_n}} \leq \frac{s}{\sqrt{W_n}}) \\ &= \Phi(\frac{s}{\sqrt{W_n}}) \\ &\to \frac{1}{2}. \end{aligned}$$

Therefore the cdf of $\frac{S_n}{n}$ converges to a function that assigns $\frac{1}{2}$ to any value on the real line. This function is obviously not a cdf.

It is possible to find a function $\alpha(n)$ that when divided into S_n, will cause the quotient to converge in probability. Note that $\forall \epsilon > 0$,

$$P\left(\left|\frac{S_n}{\alpha(n)}\right| \geq \epsilon\right) = P\left(\left|\frac{S_n}{\sqrt{V_n}}\right| \geq \frac{|\alpha(n)|\epsilon}{\sqrt{V_n}}\right).$$

Since $\frac{S_n}{\sqrt{V_n}} \sim N(0,1)$, the above expression will tend to zero iff $\frac{|\alpha(n)|\epsilon}{\sqrt{V_n}} \to \infty$. This will be true for any $\alpha(n) = o(n^k)$, where $k > \frac{3}{2}$.

It is also possible to find a function $\beta(n)$ that when divided into S_n, will cause the quotient to have a limiting normal distribution. Since we already know that $S_n \sim N(0, V_n)$, all that is needed is to find a $\beta(n)$ that is exactly of order $n^{\frac{3}{2}}$. This will cause the variance $\frac{V_n}{\beta^2(n)}$ of $\frac{S_n}{\beta(n)}$ to tend to a finite nonzero limit. $\beta(n)$ is referred to as a *variance-stabilizing transform.*

4.(a).

It is possible to prove that $m_n \xrightarrow{p} E[X_k] = 0$ (i.e., you should try it as a useful exercise).