Econ 240A: Problem Set 1
Solutions to Selected Problems from Chapter 1

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1.

Note that without insurance, Cab's optimal decision when receiving a forecast of early rainfall is to harvest early, which gives him a revenue net of the cost of purchasing the forecast of \$ 29000. When the forecast is for late rainfall, Cab will harvest late and earn an expected net revenue of $\frac{1}{5}\$10000 + \frac{4}{5}\$40000 - \$1000 = \$33000$. The expected payout of the insurance policy assuming that Cab behaves as if there were no insurance would be $\frac{1}{2}(0) + \frac{1}{2}(\frac{1}{5}(\$17900 + \frac{4}{5}(0)) = \$1790$. This is greater than the cost of the policy, so the insurance scheme is not actuarially fair if we assume that it does not alter Cab's decisions.

If Cab buys the insurance, he has

$$E[\text{Profit(Harvest Late)}|\text{Late Forecast}] = \frac{1}{5}(\$27900) + \frac{4}{5}(\$40000) = \$37580$$

and

$$E[\text{Profit(Harvest Late)}|\text{Early Forecast}] = \frac{3}{5}(\$27900) + \frac{2}{5}(\$40000) = \$32740.$$

Since $E[\text{Profit(Harvest Early)}] = \$30000$ Cab will always harvest late if he has insurance.

With insurance Cab's profit net of the cost of the forecast and the insurance is

$$\frac{1}{2}(\$32740) + \frac{1}{2}(\$37580) - \$2000 = \$33160 > \$31000,$$

so insurance will in fact be purchased.

2.

We have

$$\begin{aligned}
P[\text{Wet}|\text{Bad}] &= \frac{.15}{.2} = .75 \\
P[\text{Wet}|\text{Poor}] &= \frac{.15}{.3} = .5 \\
P[\text{Wet}|\text{Fair}] &= \frac{.1}{.3} = \frac{1}{3} \\
P[\text{Wet}|\text{Good}] &= 0
\end{aligned}$$

and

$$\begin{aligned}
E[\text{Profit(early)}] &= \$30000 \\
E[\text{Profit(late)}|\text{Bad}] &= \frac{3}{4}(\$10000) + \frac{1}{4}(\$40000) = \$17500 \\
E[\text{Profit(late)}|\text{Poor}] &= \$25000 \\
E[\text{Profit(late)}|\text{Fair}] &= \$30000 \\
E[\text{Profit(late)}|\text{Good}] &= \$40000.
\end{aligned}$$

Cab will only unambiguously prefer to harvest late with a forecast of "Good". Therefore his net expected profit is

$$\frac{4}{5}(\$30000) + \frac{1}{5}(\$40000) - \$1500 = \$30500 < \$31000,$$

so this more detailed forecast will not be purchased.

3.

The problem of maximizing expected mean excess return subject to a constraint on the variance can be written succinctly as

$$\max_{A;\theta_1,\ldots,\theta_K \geq 0} A \sum_{k=1}^{K} \theta_k r_k$$

subject to the constraints

$$\begin{aligned}
A^2 \sum_{k=1}^{K} \sum_{j=1}^{K} \theta_k \theta_j \sigma_{kj} &= c, \\
\sum_{k=1}^{K} \theta_k &= 1.
\end{aligned}$$

To solve this we define the Lagrangian

$$\mathcal{L}(A;\theta_1,\ldots,\theta_K) = A \sum_k \theta_k r_k - \lambda(A^2 \sum_k \sum_j \theta_k \theta_j \sigma_{kj} - c) - \mu(\sum_k \theta_k - 1).$$

The problem reduces to maximizing $\mathcal{L}$ subject to $\theta_k \geq 0 \forall k$.

By the Kuhn-Tucker theorem there exist constants ν_k, $k = 1, \ldots, K$ such that

$$\frac{\partial \mathcal{L}}{\partial \theta_k} = \nu_k,$$

where $\nu_k \leq 0 \forall k$ with equality only if $\theta_k > 0$. See Mas-Colell et al. (1995, p. 959) or Varian (1992, p. 503) for details.

Therefore the first-order conditions for the problem are

$$\sum_k \theta_k r_k = \lambda 2A \sum_k \sum_j \theta_k \theta_j \sigma_{kj},$$

$$A r_k \leq \lambda 2A^2 \sum_j \theta_j \sigma_{kj} + \mu, k = 1, \ldots, K,$$

where the second first-order condition holds with equality unless $\theta_k = 0$.

These are equivalent to the first-order conditions for the problem of minimizing variance subject to a constraint on the expected mean excess return (Equations (1) and (2) on page 11 of the text) and so solving out the FOCs yields $\mu = 0$. Therefore λ is interpreted as the marginal rate of substitution between mean and variance at the optimal portfolio.

References

[1] Mas-Colell, A., M. D. Whinston and J. R. Green (1995): *Microeconomic Theory*. New York and Oxford: Oxford University Press.

[2] Varian, H. R (1992): *Microeconomic Analysis*, Third Edition. New York: W. W. Norton.