Econ 204
Supplement to Section 3.3
The Matrix Representation of a Linear Transformation

The purpose of this note is to provide a brief treatment of quotient vector spaces, and amplify on the relationship between linear transformations and their matrix representations.

Definition 1 Given a vector space X and a vector subspace $W \subseteq X$, define an equivalence relation $\sim$ by

$$x \sim y \Leftrightarrow x - y \in W$$

Exercise 2 Show that $\sim$ is an equivalence relation.

Definition 3 We define a new vector space V/W, the *quotient* of V by W. The set of vectors in V/W is

$$\{[x] : x \in X\}$$

where we recall that $[x]$ is the equivalence class of x with respect to the equivalence relation $\sim$. In other words,

$$[x] = \{y \in X : x - y \in W\} = \{x + w : w \in W\}$$

Thus, each of the vectors is a *set*; this is a little weird at first, but try to get used to it. Now, we have to define the operations of vector addition and scalar multiplication. The definitions are

$$\begin{aligned} [x] + [y] &= [x + y] \\ \alpha[x] &= [\alpha x] \end{aligned}$$

One needs to check that these definitions make sense. $[x]$ is a set, and there are potentially many different representatives, i.e. many x' such that $[x] = [x']$; one must check that you get the same answer, independent of which representative is chosen in the definition.

Exercise 4 Show that vector addition and scalar multiplication are well-defined. In other words, if $[x] = [x']$, then

$$[x + y] = [x' + y] \text{ and } [\alpha x] = [\alpha x']$$

Theorem 5 *If* $\dim X$ *is finite, then* $\dim (X/W) = \dim X - \dim W$*.*

Theorem 6 *Let* $T \in L(X, Y)$*. Then* $\text{Im}\, T$ *is isomorphic to* $X/(\ker T)$*.*

Proof: If $\dim X$ is finite, this is a corollary of Theorem 5 above and Theorem 3.3 of de la Fuente. However, the result is true even if $\dim X = \infty$; moreover, the isomorphism is quite natural.

Let $\hat{T} : X/(\ker T) \to Y$ be defined by $\hat{T}([x]) = T(x)$. We need to check that $\hat{T}$ is well-defined. If $x \in X$,

$$\begin{aligned} x \sim x' &\Rightarrow x - x' \in \ker T \\ &\Rightarrow T(x - x') = 0 \\ &\Rightarrow T(x) - T(x') = 0 \\ &\Rightarrow T(x) = T(x') \end{aligned}$$

so $\hat{T}$ is well-defined.

If $[x] \in X/(\ker T)$, then $\hat{T}([x]) = T(x) \in \text{Im}\, T$. It is easy to check that $\hat{T}$ is linear. We need to show that $\hat{T}$ is one-to-one and onto. Suppose that $\hat{T}([x]) = \hat{T}([y])$. Then $T(x) = \hat{T}([x]) = \hat{T}([y]) = T(y)$, so $T(x - y) = T(x) - T(y) = 0$, so $x - y \in \ker T$, so $[x] = [y]$; this shows that $\hat{T}$ is one-to-one. If $y \in \text{Im}\,(T)$, then $\exists_{x \in X} T(x) = y$, so $\hat{T}([x]) = y$, so $\hat{T}$ is onto. ■

Now, we change gears and discuss further the isomporphism between $L(X, Y)$ and the vector space of matrices.

Theorem 7 (Theorem 3.5, page 132 of de la Fuente) *Let* X *and* Y *be vector spaces defined over the same field* F*, with finite dimensions* n *and* m *respectively. Then* $L(X, Y)$*, the vector space of linear transformations from* X *to* Y*, is isomorphic to* $F_{m \times n}$*, the vector space of* $m \times n$ *matrices with entries in* F*.*

As stated, the theorem is pretty obvious: the two spaces are easily seen to have the same dimension mn, and any two vector spaces of a given dimension over the same field are always isomorphic. The proof of the theorem in de la Fuente indicates that something more is going on; the coefficients in the matrix representation tell you what the linear transformation does to elements of the basis of X.

In this note, we show that still more is true. The next theorem shows that the matrix representation of the composition of two linear transformations is the matrix product of the matrix representations of the two transformations.

Theorem 8 (The Commutative Diagram Theorem) *Let X, Y, Z be finite-dimensional vector spaces with bases U, V, W respectively, and suppose $S \in L(X, Y)$, $T \in L(Y, Z)$. Then*

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

Proof: Let

$$\begin{aligned} U &= \{u_1, \ldots, u_j, \ldots u_n\} \\ V &= \{v_1, \ldots, v_i, \ldots v_m\} \\ W &= \{w_1, \ldots, w_k, \ldots w_r\} \\ Mtx_{V,U}(S) &= (\alpha_{ij}) \\ Mtx_{W,V}(T) &= (\beta_{ki}) \end{aligned}$$

From the definition of the matrix representation,

$$\begin{aligned} S(u_j) &= \sum_{i=1}^{m} \alpha_{ij} v_i \\ T \circ S(u_j) &= T(S(u_j)) \\ &= T\left(\sum_{i=1}^{m} \alpha_{ij} v_i\right) \\ &= \sum_{i=1}^{m} \alpha_{ij} T(v_i) \\ &= \sum_{i=1}^{m} \alpha_{ij} \left(\sum_{k=1}^{r} \beta_{ki} w_k\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{r} \left(\sum_{i=1}^{m} \alpha_{ij} \beta_{ki} \right) w_k \\
&= \sum_{k=1}^{r} (\alpha_j \cdot \beta_k) w_k \qquad (1)
\end{aligned}
$$

where α_j is the j^{th} column of $Mtx_{V,U}(S)$ and β_k is the k^{th} row of $Mtx_{W,V}(T)$. By the definition of the matrix representation of $T \circ S$, Equation (1) says that the (k, j) entry of $Mtx_{W,U}(T \circ S)$ is $\alpha_j \cdot \beta_k$. By the definition of matrix multiplication, $\alpha_j \cdot \beta_k$ is the (k, j) entry of $Mtx_{W,V}(T) \cdot Mtx_{V,U}(S)$. Thus,

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

■

The theorem can be summarized by the following "Commutative Diagram:"

$$
\begin{array}{ccccccc}
 & & S & & T & & \\
 & X & \rightarrow & Y & \rightarrow & Z & \\
crd_U & \updownarrow & & \updownarrow crd_V & & \updownarrow crd_W & \\
 & \mathbf{R}^n & \rightarrow & \mathbf{R}^m & \rightarrow & \mathbf{R}^r & \\
 & & Mtx_{V,U}(S) & & Mtx_{W,V}(T) & &
\end{array}
$$

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The *crd* arrows go in both directions because *crd* is an isomorphism.