Econ 204 Section 2
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Key Words

Deduction, Contraposition, Contradiction, Induction, Binary Relation, Equivalence Relation, Equivalence Class, Cardinality, Bijection, Numerically Equivalent

Section 2.1 Methods of Proof

- Deduction

  To prove $A \Rightarrow Z$, deduction goes like $A \Rightarrow B \Rightarrow \cdots \Rightarrow Y \Rightarrow Z$

- Contraposition

  To prove $A \Rightarrow Z$, contraposition is to prove $\neg Z \Rightarrow \neg A$. It goes like $\neg Z \Rightarrow \neg Y \Rightarrow \cdots \Rightarrow \neg B \Rightarrow \neg A$

- Induction

  A typical structure of proof is

  For $n = 0$ (or other initial value), show that the statement is true. This is the base step.

  For $n = k$, suppose that the statement is true. This is the inductive hypothesis.

  For $n = k + 1$, use what we get from the inductive hypothesis to show that the statement holds for the case of $n = k + 1$

  Conclude that the statement is true for all n.

- Contradiction

  To prove $A \Rightarrow Z$ by contradiction, we first suppose Z is not true. Then we check whether it leads to results that contradict with A or what we get from A.

  Contraposition can be regarded as a special case of contradiction. Contraposition means$\neg Z \Rightarrow \neg A$ so that we get results that contradicts with A.

- Definition Convergence of a Sequence of Real Numbers in Euclidean Metric Space.

  A sequence of real numbers $\{x_n\}$ converges to a real number x if $\forall \varepsilon > 0 \, \exists N(\varepsilon) \in \mathbf{N}$, for all $n > N(\varepsilon) \Rightarrow \mid x_n - x \mid < \varepsilon$. We denote it as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

  The following two examples will use this definition. We are going to learn the general definition of convergence of sequences in metric spaces in Lecture 3. However, they are pretty similar.

  Example 2.1.1 Prove the following statement by deduction:

  $\{x_n\}$ is a sequence of real numbers. If $\lim_{n\to\infty} x_n = x > 0$,then there exists $N \in \mathbf{N}$ such that $n > N \Rightarrow x_n > 0$.

  Solution:

  Since $\lim_{n\to\infty} x_n = x > 0$. The definition of convergence means $\forall \varepsilon > 0 \, \exists N(\varepsilon) \in \mathbf{N}$, for all $n > N(\varepsilon) \Rightarrow \mid x_n - x \mid < \varepsilon$. Let $\varepsilon = \frac{x}{2} > 0$. Then there exists $N(\varepsilon)$ such that $n > N(\varepsilon) \Rightarrow \mid x_n - x \mid < \varepsilon = \frac{x}{2}$. Since $\mid x_n - x \mid < \frac{x}{2} \Rightarrow \frac{x}{2} < x_n < \frac{3x}{2}$. We have $\frac{x}{2} > 0$. So there exists $N(\varepsilon)$ such that $n > N(\varepsilon) \Rightarrow x_n > 0$.

  Example 2.1.2 Prove the following statement by contradiction:

  $\{x_n\}$ is a sequence of real numbers. If every subsequence of $\{x_n\}$ converges to a real number x, then $\lim_{n\to\infty} x_n = x$.

  Solution:

  Suppose $\lim_{n\to\infty} x_n \neq x$. Since the definition of convergence means $\forall \varepsilon > 0 \, \exists \, N(\varepsilon) \in \mathbf{N}$, for all $n > N(\varepsilon) \Rightarrow \mid x_n - x \mid < \varepsilon$, the non-convergence means $\exists \, \varepsilon_0 > 0$, for all $N \in \mathbf{N}$, $\exists$ $n(N) > N$ such that $\mid x_n - x \mid \geq \varepsilon_0$. Then we can construct a subsequence $\{x_{n_k}\}$ as follows:

  Let $n_1 \geq 1$, such that $\mid x_{n_1} - x \mid \geq \varepsilon_0$.

  Let $n_2 > n_1$, such that $\mid x_{n_2} - x \mid \geq \varepsilon_0$.

  $\vdots$

Let $n_{k+1} > n_k$, such that $\mid x_{n_{k+1}} - x \mid \geq \varepsilon_0$.
So we obtain a subsequence $\{x_{n_k}\}$ such that $\lim_{n\to\infty} x_{n_k} \neq x$. Contradiction.
As you can see, when we write the negation of the definition of convergence, an easy way is to switch $\forall$(for all) and $\exists$ (there exist).
Constructing a counter example is not easy. It requires inspiration and experience.

Example 2.1.3 Prove the following statement by induction:
$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
Solution:
First, consider the base case $n = 1$. $1^2 = 1 = \frac{1(1+1)(2+1)}{6}$.
Now suppose that the statement holds for some n (the inductive hypothesis).
We want to show that it holds for $n + 1$ as well (the inductive step). By the induction hypothesis, we have

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}. \\ n(n+1)(2n+1) + 6(n+1)^2 &= (n+1)(6(n+1) + n(2n+1)) \\ &= (n+1)(3(n+2) + n(2n+4)) \\ &= (n+1)(n+2)(2(n+1)+1), \end{aligned}$$

It follows that $1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$.
We conclude, by induction, that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for any $n \geq 1$.
Induction is the easiest one since you have already got the result.
Make sure that you are familiar with how to calculate the summation of a sequence by induction.

Setion 2.2 Binary Relation

- Lecture 1 Definition 4: A **binary relation** R on a set X is a subset of $X \times X$. Write xRy as an abbreviation for $(x, y) \in R$.
- Lecture 1 Definition 5: A **binary relation** R on a set X is also an **equivalence relation** if it is
  - reflexive: for all $x \in X$, xRx.
  - symmetric: for all $x, y \in X$, xRy if and only if yRx.
  - transitive: for all $x, y, z \in X$, if xRy and yRx, then also xRz.
- Lecture 1 Definition 6: Given an **equivalence relation** R, write $[x] = \{y \in X : xRy\}$. $[x]$ is called the **equivalence class** containing x.

Example 2.2.1 Binary Relation & Equivalence Relation
The followings are three examples of a binary relation R on X. Are they equivalence relations?
Solution:
1. Suppose $X = \{1, 2, 3\}$. Let $R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$.
According to the definition, R is a subset of $X \times X$. So $R \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$. Let's represent it in a graph (the first element is represented by the horizontal axis):

3			·
2		·	·
1	·	·	·
	1	2	3

In fact, R is the binary relation $\geq$ (is weakly greater than).
Check the three conditions:
reflexive yes
symmetric no
transitive yes
So R is not an equivalence relation.
2. Suppose $X = \{1, 2, 3\}$. Let $R = \{(1, 1),\ (2, 2),\ (3, 3)\}$.
Solution:
Similarly we can represent the binary relation $=$ on X graphically as (the first element is represented by the horizontal axis):

3			·
2		·	
1	·		
	1	2	3

In fact, R is the binary relation $=$ (is equivalent to).
Check the three conditions:
reflexive yes
symmetric yes
transitive yes
So R is an equivalence relation.
3. Suppose $X = \{1, 2, 3, 4\}$. Let $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$.
Solution:
We can represent R in a graph (the first element is represented by the horizontal axis):

4			·	·
3			·	·
2	·	·		
1	·	·		
	1	2	3	4

Check the three conditions:
reflexive yes
symmetric yes
transitive yes
So R is an equivalence relation.

Section 2.3 Numerically Equivalent

- Lecture 1 Section 1.4: A function $f : A \to B$ is a **bijection** if it is
  - **one to one**, i.e. $a \neq a' \Rightarrow f(a) \neq f(a')$
  - **onto**, i.e. $\forall b \in B, \exists a \in A\, f(a) = b$
- Lecture 1 Section 1.4: Two sets A, B are **numerically equivalent** (have the same Cardinality) if there exists a **bijection** $f : A \to B$.

Example 2.3.1 Numerically Equivalent
Prove that $(0, 1)$ is numerically equivalent to **R.**
Solution:
Recall the function $\tan(x)$. The range of $\tan(x)$ is **R**. The domain of $\tan(x)$ is all real numbers excluding $\frac{\pi}{2} + n\pi$ where n is an integer. Let $f(x) = \tan(\pi x - \frac{\pi}{2})$, $x \in (0, 1)$. f is a bijection from $(0, 1)$ to **R.**

The basic way to prove set A is numerically equivalent to set B is to construct a bijection between A and B.

Coutable means being numerically equivalent to **N**. Picture proof could be useful if we want to prove or disprove that a set is countable.

Example 2.3.2 The Uncountability of the Real Numbers.

Prove that $(0, 1)$ is not countable.

Solution:

We do it by picture proof and contradiction. Suppose that $(0, 1)$ is countable. Then for every $x \in (0, 1)$ there exists a unique $n \in \mathbf{N}$ corresponding to x which is the bijection from N to $(0, 1)$. Denote it as x_n, $n = 1, 2, 3 \ldots$. Then write out each x_n in a decimal expansion in a picture such that

$x_1 = 0.a_{11}a_{12}a_{13}a_{14}...$
$x_2 = 0.a_{21}a_{22}a_{23}a_{24}...$
$x_3 = 0.a_{31}a_{32}a_{33}a_{34}...$
$x_4 = 0.a_{41}a_{42}a_{43}a_{44}...$

where each a_{ik} is a natural number between 0 and 9. Now consider the real number $x = 0.b_1b_2b_3\ldots$where each b_k is chosen such that $b_k = 1$ if $a_{kk} \neq 1$ and $b_k = 2$ if $a_{kk} = 1$. Then $x \neq x_k$ for all $k \in \mathbf{N}$ since it differs from x_k in the kth decimal place. But $x \in (0, 1)$. Contradiction. So there is no bijection from **N** to $(0, 1)$. $(0, 1)$ is not countable.

In **Example 2.3.1** we proved that $(0, 1)$ is numerically equivalent to **R.** So we can further conclude that **R** is not countable**.**