Economics 204

Lecture 9–Thursday, August 6, 2009

Section 3.3 Supplement, Quotient Vector Spaces (not in de la Fuente):

Definition 1 Given a vector space X and a vector subspace W of X, define an equivalence relation by

$$x \sim y \Leftrightarrow x - y \in W$$

Form a new vector space X/W: the set of vectors is

$$\{[x] : x \in X\}$$

where $[x]$ denotes the equivalence class of x with respect to $\sim$. Note that the vectors are *sets*; this is a little weird at first, but . . . . Define

$$\begin{aligned} [x] + [y] &= [x + y] \\ \alpha[x] &= [\alpha x] \end{aligned}$$

You should check on your own that $\sim$ is an equivalence relation and that vector addition and scalar multiplication are well-defined, i.e.

$$\begin{aligned} [x] = [x'], [y] = [y'] &\Rightarrow [x + y] = [x' + y'] \\ [x] = [x'], \alpha \in F &\Rightarrow [\alpha x] = [\alpha x'] \end{aligned}$$

Theorem 2 *If* $\dim X < \infty$, *then*

$$dim(X/W) = \dim X - \dim W$$

Theorem 3 *Let* $T \in L(X, Y)$. *Then* $\operatorname{Im} T$ *is isomorphic to* $X/\ker T$.

Proof: If $\dim X < \infty$, then $\dim X/\ker T = \dim X - \dim\ker T$ (from the previous theorem) $= \operatorname{Rank} T$ (from Theorem 11 in yesterday's lecture) $= \dim \operatorname{Im} T$, so $X/\ker T$ is isomorphic to $\operatorname{Im} T$. We shall prove that it is true in general, and that the isomorphism is natural. Define

$$\tilde{T}([x]) = T(x)$$

We need to check that this is well-defined.

$$\begin{aligned} [x] = [x'] \quad &\Rightarrow \quad x \sim x' \\ &\Rightarrow \quad x - x' \in \ker T \\ &\Rightarrow \quad T(x - x') = 0 \\ &\Rightarrow \quad T(x) = T(x') \end{aligned}$$

so $\tilde{T}$ is well-defined. Clearly, $\tilde{T} : X/\ker T \to \operatorname{Im} T$. It is easy to check that $\tilde{T}$ is linear, so $\tilde{T} \in L(X/\ker T, \operatorname{Im} T)$.

$$\begin{aligned} \tilde{T}([x]) = \tilde{T}([y]) \quad &\Rightarrow \quad T(x) = T(y) \\ &\Rightarrow \quad T(x - y) = 0 \\ &\Rightarrow \quad x - y \in \ker T \\ &\Rightarrow \quad x \sim y \\ &\Rightarrow \quad [x] = [y] \end{aligned}$$

so $\tilde{T}$ is one-to-one.

$$\begin{aligned} y \in \operatorname{Im} T \quad &\Rightarrow \quad \exists_{x \in X}\ T(x) = y \\ &\Rightarrow \quad \tilde{T}([x]) = y \end{aligned}$$

so $\tilde{T}$ is onto, hence $\tilde{T}$ is an isomorphism. ■

Example: Consider $T \in L(\mathbf{R}^3, \mathbf{R}^2)$ defined by

$$T(x, y, z) = (x, y)$$

Then $\ker T = \{(x, y, z) \in \mathbf{R}^3 : x = y = 0\}$ is the z-axis. Given (x, y, z), the equivalence class $[(x, y, z)]$ is just the line through $(x, y, 0)$ parallel to the z-axis. $\tilde{T}([(x, y, z)]) = T(x, y, z) = (x, y)$.

Back to de la Fuente:

Every real vector space X with dimension n is isomorphic to $\mathbf{R}^n$. What's the isomorphism?

Definition 4 Fix any Hamel basis $V = \{v_1, \ldots, v_n\}$ of X. Any $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} \beta_j v_j$$

(here, we allow $\beta_j = 0$). Generally, vectors are represented as column vectors, not row vectors.

$$crd_V(x) = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbf{R}^n$$

$crd_V(x)$ is the vector of coordinates of x with respect to the basis V.

$$crd_V(v_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad crd_V(v_2) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad crd_V(v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

crd_V is an isomorphism from X to $\mathbf{R}^n$.

Matrix Representation of a Linear Transformation

Definition 5 Suppose $T \in L(X, Y)$, $\dim X = n$, $\dim Y = m$. Fix bases

$$\begin{aligned} V &= \{v_1, \ldots, v_n\} \text{ of } X \\ W &= \{w_1, \ldots, w_m\} \text{ of } Y \end{aligned}$$

$T(v_j) \in Y$, so

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$$

Define

$$Mtx_{W,V}(T) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

Notice that the columns are the coordinates (expressed with respect to W) of $T(v_1), \ldots, T(v_n)$.

Observe

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \vdots \\ \alpha_{m1} \end{pmatrix}$$

so

$$\begin{aligned} Mtx_{W,V}(T) \cdot crd_V(v_j) &= crd_W(T(v_j)) \\ \forall_{x \in X}\ Mtx_{W,V}(T) \cdot crd_V(x) &= crd_W(T(x)) \end{aligned}$$

Multiplying a vector by a matrix does two things:

- Computes the action of T
- Accounts for the change in basis

Example: $X = Y = \mathbf{R}^2$, $V = \{(1,0),(0,1)\}$, $W = \{(1,1),(-1,1)\}$, $T = id$, $T(x) = x$,

$$Mtx_{W,V}(T) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$Mtx_{W,V}(T)$ is the matrix which *changes basis* from V to W. How do we compute it?

$$v_1 = (1,0) \quad = \quad \alpha_{11}(1,1) + \alpha_{21}(-1,1)$$

$$
\begin{aligned}
\alpha_{11} - \alpha_{21} &= 1 \\
\alpha_{11} + \alpha_{21} &= 0 \\
2\alpha_{11} &= 1, \alpha_{11} = \frac{1}{2} \\
\alpha_{21} &= -\frac{1}{2} \\
v_2 = (0,1) &= \alpha_{12}(1,1) + \alpha_{22}(-1,1) \\
\alpha_{12} - \alpha_{22} &= 0 \\
\alpha_{12} + \alpha_{22} &= 1 \\
2\alpha_{12} &= 1, \alpha_{12} = \frac{1}{2} \\
\alpha_{22} &= \frac{1}{2}
\end{aligned}
$$

$$Mtx_{W,V}(id) = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Theorem 6 (3.5') *Let X and Y be vector spaces over the same field F, $\dim X = n$, $\dim Y = m$, bases*

$$V = \{v_1, \ldots, v_n\} \text{ for } X$$

$$W = \{w_1, \ldots, w_m\} \text{ for } Y$$

Then

$$Mtx_{W,V} \in L(L(X,Y), F_{m\times n})$$

$Mtx_{W,V}$ is an isomorphism from $L(X,Y)$ to $F_{m\times n}$, the vector space of $m \times n$ matrices over F.

Theorem 7 ((From Handout)) *Let X, Y, Z be finite-dimensional vector spaces with bases U, V, W respectively, $S \in L(X,Y)$, $T \in L(Y,Z)$. Then*

$$Mtx_{W,V}(T) \cdot Mtx_{V,U}(S) = Mtx_{W,U}(T \circ S)$$

i.e. matrix multiplication corresponds via the isomorphism to composition of linear transformations. Note that $Mtx_{W,V}$ is a function from $L(X,Y)$ to the space of $m \times n$ matrices, while $Mtx_{W,V}(T)$ is an $m \times n$ matrix.

Proof: See handout.■

The theorem can be summarized by the following "Commutative Diagram:"

$$\begin{array}{cccccc} & & S & & T & \\ & X & \rightarrow & Y & \rightarrow & Z \\ crd_U & \updownarrow & & \updownarrow crd_V & & \updownarrow crd_W \\ & \mathbf{R}^n & \rightarrow & \mathbf{R}^m & \rightarrow & \mathbf{R}^r \\ & & Mtx_{V,U}(S) & & Mtx_{W,V}(T) & \end{array}$$

We say the diagram commutes because you get the same answer any way you go around the diagram (in directions allowed by the arrows). The *crd* arrows go in both directions because *crd* is an isomorphism.

Section 3.5, Change of Basis, Similarity

Let X be a finite-dimensional vector space with basis V. If $T \in L(X,X)$ it is customary to use the same basis in the domain and range:

Definition 8

$$Mtx_V(T) \text{ denotes } Mtx_{V,V}(T)$$

Question: If W is another basis for X, how are $Mtx_V(T)$ and $Mtx_W(T)$ related?

$$\begin{aligned} & Mtx_{V,W}(id) \cdot Mtx_W(T) \cdot Mtx_{W,V}(id) \\ = \;& Mtx_{V,W}(id) \cdot Mtx_{W,V}(T \circ id) \\ = \;& Mtx_{V,V}(id \circ T \circ id) \end{aligned}$$

$$= Mtx_V(T)$$

$$Mtx_{V,W}(id) \cdot Mtx_{W,V}(id)$$

$$= Mtx_{V,V}(id)$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$Mtx_V(T) = P^{-1} Mtx_W(T) P$$

where

$$P = Mtx_{W,V}(id)$$

is a change of basis matrix. On the other hand, if P is invertible, then P is a change of basis matrix (see handout).

Definition 9 Square matrices A, B are *similar* if

$$A = P^{-1}BP$$

for some invertible matrix P.

Theorem 10 *Suppose that X is finite-dimensional.*

- *If $T \in L(X, X)$ and U, W are any two bases of X, then $Mtx_W(T)$ and $Mtx_U(T)$ are similar.*
- *Conversely, given similar matrices A, B with $A = P^{-1}BP$ and any basis U, there is a basis W and $T \in L(X, X)$ such that*

$$B = Mtx_U(T)$$

$$A = Mtx_W(T)$$

$$P = Mtx_{U,W}(id)$$

$$P^{-1} = Mtx_{W,U}(id)$$

Proof: See Handout on Diagonalization and Quadratic Forms.■

Section 3.6: Eigenvalues and Eigenvectors

De la Fuente defines eigenvalues and eigenvectors of a matrix. Here, we define eigenvalues and eigenvectors of a linear transformation and show that λ is an eigenvalue of T if and only if λ is an eigenvalue for some matrix representation of T if and only if λ is an eigenvalue for every matrix representation of T.

Definition 11 Let X be a vector space and $T \in L(X, X)$. We say that λ is an *eigenvalue* of T and $v \neq 0$ is an *eigenvector corresponding to* λ if $T(v) = \lambda v$.

Theorem 12 (Theorem 4 in Handout) *Let X be a finite-dimensional vector space, and U any basis. Then λ is an eigenvalue of T if and only if λ is an eigenvalue of $Mtx_U(T)$. v is an eigenvector of T corresponding to λ if and only if $crd_U(v)$ is an eigenvector of $Mtx_U(T)$ corrresponding to λ.*

Proof: By the Commutative Diagram Theorem,

$$\begin{aligned} T(v) = \lambda v \quad &\Leftrightarrow \quad crd_U(T(v)) = crd_U(\lambda v) \\ &\Leftrightarrow \quad Mtx_U(T)(crd_U(v)) = \lambda(crd_U(v)) \end{aligned}$$

■ *Computing eigenvalues and eigenvectors:*

Suppose $\dim X = n$; let I be the $n \times n$ identity matrix. Given $T \in L(X, X)$, fix a basis U and let

$$A = Mtx_U(T)$$

Find the eigenvalues of T by computing the eigenvalues of A:

$$\begin{aligned} Av = \lambda v &\Leftrightarrow (A - \lambda I)v = 0 \\ &\Leftrightarrow (A - \lambda I) \text{ is not invertible} \\ &\Leftrightarrow \det(A - \lambda I) = 0 \end{aligned}$$

We have the following facts:

- If $A \in \mathbf{R}_{n \times n}$,

  $$f(\lambda) = \det(A - \lambda I)$$

  is an n^{th} degree polynomial in λ with real coefficients; it is called the *characteristic polynomial* of A.

- f has n roots in $\mathbf{C}$, counting multiplicity:

  $$f(\lambda) = (\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_n)$$

  where $c_1, \ldots, c_n \in \mathbf{C}$ are the eigenvalues; the c_j's are not necessarily distinct.

- the roots which are not real come in conjugate pairs:

  $$f(a + bi) = 0 \Leftrightarrow f(a - bi) = 0$$

- if $\lambda = c_j \in \mathbf{R}$, there is a corresponding eigenvector in $\mathbf{R}^n$.

- if $\lambda = c_j \notin \mathbf{R}$, the corresponding eigenvectors are in $\mathbf{C}^n \setminus \mathbf{R}^n$.

Diagonalization

Definition 13 Suppose X is finite-dimensional with basis U. Given a linear transformation $T \in L(X, X)$, let

$$A = Mtx_U(T)$$

We say that A *can be diagonalized* if there is a basis W for X such that $Mtx_W(T)$ is diagonal, i.e.

$$Mtx_W(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Notice that the eigenvectors of $Mtx_W(T)$ are exactly the standard basis vectors of $\mathbf{R}^n$. But w_j is an eigenvector of T for λ_j if and only if $crd_W(w_j)$ is an eigenvector of $Mtx_W(T)$, and $crd_W(w_j)$ is the j^{th} standard basis vector of $\mathbf{R}^n$, so $W = \{w_1, \ldots, w_n\}$ where w_j is an eigvenvector corresponding to λ_j.

Then the action of T is clear: it stretches each basis element w_i by the factor λ_i.

Theorem 14 (6.7') *Let X be an n-dimensional vector space, $T \in L(X, X)$, U any basis of X, and $A = Mtx_U(T)$. Then the following are equivalent:*

- *A can be diagonalized*
- *there is a basis W for X consisting of eigenvectors of T*
- *there is a basis V for $\mathbf{R}^n$ consisting of eigenvectors of A*

Proof: Use de la Fuente's Theorem 6.7 and Theorem 4 from the Handout.■

Theorem 15 (6.8') *Let X be a vector space, $T \in L(X, X)$.*

- *If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors $v_1, \ldots, v_m$, then $\{v_1, \ldots, v_m\}$ is linearly independent.*

- *If* $\dim X = n$ *and* T *has* n *distinct eigenvalues, then* X *has a basis consisting of eigenvectors of* T*; consequently, if* U *is any basis of* X*, then* $Mtx_U(T)$ *is diagonalizable.*

Proof: This is an adaptation of the proof of Theorem 6.8 in de la Fuente.■