Economics 204
Lecture 7–Tuesday, August 4, 2009
Revised 8/5/09, Revisions indicated by ** and Sticky Notes

Note: In this set of lecture notes, $\bar{A}$ refers to the closure of A.

Section 2.9, Connected Sets

Definition 1 Two sets A, B in a metric space are *separated* if

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

A set in a metric space is *connected* if it cannot be written as the union of two nonempty separated sets.

***Remark:* In other texts, you will see the following equivalent definition: A set Y in a metric space X is connected if there do not exist open sets A and B such that $A \cap B = \emptyset$, $Y \subseteq A \cup B$ and $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$.

Example: $[0, 1)$ and $[1, 2]$ are disjoint but not separated:

$$\overline{[0,1)} \cap [1,2] = [0,1] \cap [1,2] = \{1\} \neq \emptyset$$

$[0, 1)$ and $(1, 2]$ are separated:

$$\begin{aligned} \overline{[0,1)} \cap (1,2] &= [0,1] \cap (1,2] = \emptyset \\ [0,1) \cap \overline{(1,2]} &= [0,1) \cap [1,2] = \emptyset \end{aligned}$$

Note that $d([0, 1), (1, 2]) = 0$ even though the sets are separated. Note that separation does *not* require that $\bar{A} \cap \bar{B} = \emptyset$.

$$[0,1) \cup (1,2]$$

is not connected.

Theorem 2 (9.2) *A set S of real numbers is connected if and only if it is an interval, i.e. given $x, y \in S$ and $z \in (x, y)$, then $z \in S$.*

Proof: First, we show that S connected implies that S is an interval. We do this by proving the contrapositive: if S is not an interval, it is not connected. If S is not an interval, find

$$x, y \in S,\ x < z < y,\ z \notin S$$

Let

$$A = S \cap (-\infty, z),\ B = S \cap (z, \infty)$$

Then

$$\begin{aligned}
\bar{A} \cap B &\subseteq \overline{(-\infty, z)} \cap (z, \infty) = (-\infty, z] \cap (z, \infty) = \emptyset \\
A \cap \bar{B} &\subseteq (-\infty, z) \cap \overline{(z, \infty)} = (-\infty, z) \cap [z, \infty) = \emptyset \\
A \cup B &= (S \cap (-\infty, z)) \cup (S \cap (z, \infty)) \\
&= S \setminus \{z\} \\
&= S \\
x &\in A, \text{ so } A \neq \emptyset \\
y &\in B, \text{ so } B \neq \emptyset
\end{aligned}$$

so S is not connected. We have shown that if S is not an interval, then S is not connected; therefore, if S is connected, then S is an interval.

Now, we need to show that if S is an interval, it is connected. This is much like the proof of the Intermediate Value Theorem. See de la Fuente for the details.■

Theorem 3 (9.3) *Let X be a metric space, $f : X \to Y$ continuous. If C is a connected subset of X, then $f(C)$ is connected.*

Proof: This is problem 5(b) on Problem Set 3. The idea is in the diagram. Prove the contrapositive: if $f(C)$ is not connected, then C is not connected. ■

Corollary 4 (Intermediate Value Theorem) *If $f : [a, b] \to \mathbf{R}$ is continuous, and $f(a) < d < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.*

Proof: This is our third, and slickest, proof of the Intermediate Value Theorem. It is short because a substantial part of the proof was incorporated into the proof that $C \subseteq \mathbf{R}$ is connected if and only if C is an interval, and the proof that if C is connected, then $f(C)$ is connected. Here's the proof: $[a, b]$ is an interval, so $[a, b]$ is connected, so $f([a, b])$ is connected, so $f([a, b])$ is an interval. $f(a) \in f([a, b])$, and $f(b) \in f([a, b])$, and $d \in [f(a), f(b)]$; since $f([a, b])$ is an interval, $d \in f([a, b])$, i.e. there exists $c \in [a, b]$ such that $f(c) = d$. Since $f(a) < d < f(b)$, $c \neq a$, $c \neq b$, so $c \in (a, b)$. ■

Read on your own the material on arcwise-connectedness. Please note the discussion in the Corrections handout.

Section 2.10: Read this on your own.

Section 2.11: Continuity of Correspondences in E^n

Definition 5 A *correspondence* $\Psi : X \to Y$ is a function from X to 2^Y.

Remark 6 See Item 1 on the Corrections handout. De la Fuente's gives two inequivalent definitions of a correspondence on page 23. The first agrees with the definition we just gave, while the second requires that for all $x \in X$, $\Psi(x) \neq \emptyset$. In asserting the equivalence of the two definitions, he seems to believe, erroneously, that $\emptyset \notin 2^Y$. In the literature, you will find the term correspondence defined in both ways, so you should check what any given author means by the term. In these lectures, we do *not* impose the requirement that $\Psi(x) \neq \emptyset$, since it will be convenient in Lecture 11 to consider a correspondence such that $\Psi(x) = \emptyset$ for some values of

x. If $\Psi(x) \neq \emptyset$ for all x, we will say that Ψ is "nonempty-valued."

We want to talk about continuity of correspondences in a way analogous to continuity of functions. One way a function may be discontinuous at a point x_0 is that it "jumps upward at the limit:"

$$\exists_{x_n \to x_0} f(x_0) > \limsup f(x_n)$$

It could also "jump downward at the limit:"

$$\exists_{x_n \to x_0} f(x_0) < \liminf f(x_n)$$

In either case, it doesn't matter whether the sequence x_n approaches x_0 from the left or the right (or both).
What should it mean for a *set* to "jump down" at the limit x_0? It should mean the set suddenly gets smaller, i.e. it "implodes in the limit;" in other words there is a sequence $x_n \to x_0$ and points $y_n \in \Psi(x_n)$ that are far from every point of $\Psi(x_0)$. The set "jumps up" should mean that that the set suddenly gets bigger, i.e. it "explodes in the limit;" in other words, there is a point y in $\Psi(x_0)$ and a sequence $x_n \to x$ such that y is far from every point of $\Psi(x_n)$.

Remark 7 *Caution*: De la Fuente uses the term "explode" and "implode," but not "at the limit." For him, a set explodes if it suddenly gets bigger, which agrees with our use; however, instead of looking at whether the set explodes *at the limit* x_0, he looks instead at whether the set explodes *as you move slightly away from the limit* x_0, which is equivalent to *imploding at the limit.* Our approach follows the more conventional use in the literature, while de la Fuente's use is the opposite.

Remark 8 De la Fuente defines correspondences only with domain equalling a Euclidean space. In fact, we need correspondence defined on subsets of Euclidean space, so we need to modify his definition.

Definition 9 Let $X \subseteq \mathbf{E}^n$, $Y \subseteq \mathbf{E}^m$. Suppose $\Psi : X \to Y$ is a correspondence.

- Ψ is *upper hemicontinuous (uhc)* at $x_0 \in X$ if, for every open set $V \supseteq \Psi(x_0)$, there is an open set U with $x_0 \in U$ such that

  $$\Psi(x) \subseteq V \text{ for every } x \in U \cap X$$

  This says Ψ doesn't "implode in the limit" at x_0;

- Ψ is *lower hemicontinuous (lhc)* at $x_0 \in X$ if, for every open set V such that $\Psi(x_0) \cap V \neq \emptyset$, there is an open set U with $x_0 \in U$ such that

  $$\Psi(x) \cap V \neq \emptyset \text{ for every } x \in U \cap X$$

  This says Ψ doesn't "explode in the limit" at x_0;

- Ψ is *continuous* at $x_0 \in X$ if it is both uhc and lhc at x_0.

- Ψ is *closed* (*has closed graph*) if its graph

  $$\{(x, y) : y \in \Psi(x)\} \text{ is a closed subset of } X \times \mathbf{E}^m$$

Note that the definition of lower hemicontinuity does not just replace $\Psi(x_0) \subseteq V$ in the definition of upper hemicontinuity with $V \subseteq \Psi(x_0)$; indeed, we will be very interested in correspondences in which $\Psi(x)$ has empty interior, so there will often be no open sets V such that $V \subseteq \Psi(x_0)$. Unfortunately, correspondences that arise in Economics are rarely continuous. The two most important concepts are upper hemicontinuity and closed graph; we will focus on these. See the drawings on the previous page.

Example: Consider the correspondence

$$\Psi(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in (0,1] \\ \{0\} & \text{if } x = 0 \end{cases}$$

$\Psi(0) = \{0\}$. Let $V = (-0.1, 0.1)$. Then $\Psi(0) \subset V$, but no matter how close x is to 0,

$$\Psi(x) = \left\{\frac{1}{x}\right\} \not\subseteq V$$

so Ψ is not uhc at 0. However, note that Ψ has closed graph.
Example: Consider the correspondence

$$\Psi(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in (0,1] \\ \mathbf{R}_+ & \text{if } x = 0 \end{cases}$$

$\Psi(0) = [0, \infty)$, so any $V \supseteq \Psi(0)$ contains $\Psi(x)$ for all x. Thus, Ψ is uhc, and has closed graph.

Theorem 10 *Let $X \subseteq E^n$, $Y \subseteq E^m$, $f : X \to Y$ a function. Let $\Psi(x) = \{f(x)\}$ for all $x \in X$. Then $\Psi(x)$ is uhc if and only if f is continuous.*

Proof: Suppose Ψ is uhc. We consider the metric spaces (X, d) and (Y, d), where d is the Euclidean metric. Fix V open in Y. Then

$$\begin{aligned} f^{-1}(V) &= \{x \in X : f(x) \in V\} \\ &= \{x \in X : \Psi(x) \subseteq V\} \end{aligned}$$

Thus, f is continuous if and only if $f^{-1}(V)$ is open in X for each open V in Y, if and only if $\{x \in X : \Psi(x) \subseteq V\}$ is open in X for each open V in Y, if and only if Ψ is uhc (as an exercise, think through why this last equivalence holds).■

Definition 11 Suppose $X \subseteq E^m$, $Y \subseteq E^n$. A correspondence $\Psi : X \to Y$ is called *closed-valued* if $\Psi(x)$ is a closed subset of E^n for all x; Ψ is called *compact-valued* if $\Psi(x)$ is compact for all x.

The definition of upper hemicontinuity doesn't handle very well correspondences which are not closed-valued; it is not hard to construct examples of pairs of correspondences which look equally well-behaved (or ill-behaved) in which one of the correspondences is uhc and the other is not. However, for closed-valued correspondences, things are much better.

Theorem 12 (Not in de la Fuente) *Suppose* $X \subseteq \mathbf{E}^n$ *and* $Y \subseteq \mathbf{E}^m$, *and* $\Psi : X \to Y$ *is a correspndence.*

- *If* Ψ *is closed-valued and uhc, then* Ψ *has closed graph.*
- *If* $**\Psi$ *has closed graph and there is an open set* X *with* $x_0 \in X$ *and a compact set* Z *such that* $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$, *then* Ψ *is uhc at* x_0.

Proof: Suppose Ψ is closed-valued and uhc. If Ψ does not have closed graph, we can find a sequence $(x_n, y_n) \to (x_0, y_0)$, where (x_n, y_n) lies in the graph of Ψ (so $y_n \in \Psi(x_n)$) but (x_0, y_0) does not lie in the graph of Ψ (so $y_0 \notin \Psi(x_0)$). Since Ψ is closed-valued, $\Psi(x_0)$ is closed; since $y_0 \notin \Psi(x_0)$, there is some $\varepsilon > 0$ such that $\Psi(x_0) \cap B_{2\varepsilon}(y_0) = \emptyset$, so $\Psi(x_0) \subseteq \mathbf{E}^n \setminus B_\varepsilon[y_0]$. Let $V = \mathbf{E}^n \setminus B_\varepsilon[y_0]$; since V is the complement of a closed set, V is open, and it contains $\Psi(x_0)$. Since Ψ is uhc, there is an open set U with $x_0 \in U$ such that $x \in U \cap X \Rightarrow \psi(x) \subseteq V$. Since $(x_n, y_n) \to (x_0, y_0)$, $x_n \in U$ for n sufficiently large, so $y_n \in \Psi(x_n) \subseteq V$, so $|y_n - y_0| \geq \varepsilon$, which shows that $y_n \not\to y_0$, so $(x_n, y_n) \not\to (x_0, y_0)$, a contradiction that shows that Ψ is closed-graph.

**Now, suppose Ψ has closed graph and there is an open set W with $x_0 \in U$ and a compact set Z such that $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$. Since Ψ is closed-graph, it is closed-valued. Let V be any open set such that $V \supseteq \Psi(x_0)$. We need to show there exists an open set U with $x_0 \in U$ such that $x \in U \cap X \Rightarrow \Psi(x) \subseteq V$. If not, we can find a sequence $x_n \to x_0$ and $y_n \in \Psi(x_n)$ such that $y_n \notin V$. Since $x_n \to x_0$, $x_n \in W \cap X$ and thus $\psi(x_n) \subseteq Z$ for n sufficiently large. Since Z is compact, we can find a convergent subsequence $y_{n_k} \to y'$. Then $(x_{n_k}, y_{n_k}) \to (x_0, y')$; since Ψ has closed graph, $y' \in \Psi(x_0)$, so $y' \in V$. Since V is open, $y_{n_k} \in V$ for k sufficiently large, a contradiction. Thus, Ψ is uhc at x_0. ■

Theorem 13 (11.2) *Suppose $X \subseteq \mathbf{E}^n$ and $Y \subseteq \mathbf{E}^m$. A compact-valued correspondence $\Psi : X \to Y$ is uhc at $x_0 \in X$ if and only if, for every sequence $x_n \to x_0$, $\{x_n\} \subseteq X$, and every sequence $\{y_n\}$ such that $y_n \in \Psi(x_n)$, there is a convergent subsequence $\{y_{n_k}\}$ such that $\lim y_{n_k} \in \Psi(x_0)$.*

Proof: See de la Fuente.■

Remark 14 I don't find the preceding sequential characterization of uhc to be very useful or intuitive, so I recommend that you bite the bullet and master the open set definition. However, the following sequential characterization of lhc is intuitive; it says that for any $y_0 \in \Psi(x_0)$ and any x sufficiently close to x_0, we may find $y \in \Psi(x)$ such that y is close to y_0.

Theorem 15 (11.3) *A correspondence $\Psi : X \to Y$ is lhc at $x_0 \in X$ if and only if, for every sequence $x_n \to x_0$, $\{x_n\} \subseteq X$, and every $y_0 \in \Psi(x_0)$, there exists a companion sequence y_n with $y_n \in \Psi(x_n)$ such that $y_n \to y_0$.*

Proof: See de la Fuente.■