Economics 201B

Handout on Core Convergence

Core convergence theorems assert that, for economies with a large number of agents, core allocations are “approximately competitive.” The particular sense of approximate competitiveness depends on our assumptions, particularly our assumptions about preferences.

There are three motivations for the study of the core. The first two relate to what the core convergence results tell us about Walrasian equilibrium, and are normative in character. The fact that Walrasian allocations lie in the core is an important strengthening of the First Welfare Theorem. This is a strong stability property of Walrasian equilibrium: no group of individuals would choose to upset the equilibrium by recontracting among themselves. It has a further normative significance. If we are satisfied that the distribution of initial endowments has been done in an equitable manner, no group can object that it is treated unfairly at a core allocation. Since Walrasian allocations lie in the core, they possess this desirable group fairness property. This strengthening of the First Welfare Theorem requires no assumptions on the economy beyond those limited assumptions required for the First Welfare Theorem itself.

The second motivation concerns the relationship of the core convergence theorems to the Second Welfare Theorem, which asserts, under appropriate hypotheses, that any Pareto optimal allocation is a Walrasian equilibrium with transfers. The core convergence theorems assert that core allocations of large economies are *nearly* Walrasian *without* any transfers.[1] This is a strong “unbiasedness” property of Walrasian equilibrium: if a social planner were to insist that only Walrasian outcomes were to be permitted, that insistence by itself would not substantially narrow the range of possible outcomes beyond the narrowing that occurs in the core. The insistence would have no hidden implications for the welfare of different groups beyond whatever equity issues arise in the initial endowment distribution. Indeed, assuming that the distribution of endowments is equitable, any allocation that is far

[1] One version of the core convergence theorem (which we do not present here) states that core allocations can be realized as *exact* Walrasian equilibrium with *small* income transfers.

from being Walrasian will not be in the core, and hence will treat some group unfairly.

One should be cautious about interpreting the support for Walrasian equilibrium provided by the two arguments as supporting the desirability of allowing the "free market" to operate. Implicit in the definition of Walrasian equilibrium is the notion that economic agents act as price-takers. If this assumption were false, then the theoretical advantages of Walrasian allocations would shed little light on the policy issue of whether market or planned economies produce more desirable outcomes. The fact that prices are used to equate supply and demand does not guarantee that the result is Walrasian: an agent possessing market power may choose to supply quantities different from the competitive supply for the prevailing price, thereby altering that price and leading to an outcome that is not Pareto optimal. This positive issue, whether we expect the allocations produced by the market mechanism to exhibit price-taking behavior, provides the third motivation for the core convergence results.

Edgeworth [4], criticizing Walras [5], took the view that the core, rather than the set of Walrasian equilibria, was the best description of the possible allocations that the market mechanism could produce. In particular, the definition of the core does not impose the assumption of price-taking behavior made by Walras. Furthermore, if any allocation not in the core arose, some group would find it in its interests to recontract. Edgeworth thus argues that the core is the significant positive equilibrium concept.

Taking Edgeworth's point of view, a core convergence theorem can be viewed as a justification of the price-taking assumption. The theorem stated below indicates that, at a core allocation, trade occurs almost at a single price. Someone who tries to bargain with other agents for a more favorable price is unable to do so, since there will be a coalition that can block the resulting allocation. The exploitation of market power gives rise to little change in the outcome.[2]

Core convergence theorems thus provides a positive argument in favor of the price-taking assumption. Note however that the size of the endowments enters the bound in the theorem in an important way. Whether the core convergence theorems can be viewed as providing support for the price-taking

[2]This argument is more compelling when, under stronger assumptions, one obtains stronger core convergence conclusions.

assumption in a given real economy depends on the relationship of the distribution of endowments to the number of agents. Edgeworth's view was that the presence of firms, unions and other large economic units makes the core substantially larger than the set of Walrasian equilibria, a view the author shares.

The following theorem tell us that, given a Pareto optimum x, we can find a price vector p such that (p, x) nearly satisfies the definition of a Walrasian equilibrium. Note that if the bound on the right hand side of Equation (1) were zero, (p, x) would be a Walrasian quasi-equilibrium. If there are many more agents than goods, and the endowments are not too large, the bound will be small. The result is taken from E. Dierker [3] and Anderson [1]. The assumptions on the economy are extremely limited; in particular, convexity of preferences is not assumed. Indeed, assuming convexity of preferences does not make the result any easier to prove; the *definition* of the core introduces a nonconvexity into the argument, essentially because an individual may be excluded or included in a potential blocking coalition. Stronger hypotheses allow one to prove stronger conclusions.

For a survey of core convergence results, see Anderson [2].

Theorem 1 *Suppose we are given an exchange economy with L commodities, I agents, and preferences $\succ_1, \ldots, \succ_I$ satisfying weak monotonicity (if $x \gg y$, then $x \succ_i y$) and the following* free disposal *condition:*

$$x \gg y,\ y \succ_i z \quad \Rightarrow x \succ_i z.$$

If x is in the core, then there exists $p \in \Delta$ such that

$$\begin{aligned} \frac{1}{I}\sum_{i=1}^{I} & \quad (|p \cdot (x_i - \omega_i)| + |\inf\{p \cdot (y - x_i) : y \succ_i x_i\}|) \\ & \leq \frac{6L}{I}\max\{\|\omega_1\|_\infty, \ldots \|\omega_I\|_\infty\} \end{aligned} \tag{1}$$

where $\|x\|_\infty = \max\{|x_1|, \ldots, |x_L|\}$.

The proof involves the following main steps, which parallel those in the proof of the Second Welfare Theorem.

1. Suppose x is in the core. Define $B_i = \{y - \omega_i : y \succ_i x_i\} \cup \{0\}$, $B = \sum_{i=1}^{I} B_i$. Note that B_i is *not* convex, even if $\succ_i$ is a convex

preference. If $Y \in B$, then $Y = \sum_{i=1}^{I} y_i$, with $y_i \in B_i$. If $Y \ll 0$, it is easy to see that the coalition $S = \{i : y_i \succ_i x_i\}$ can block the allocation x. Thus, since x is in the core, $B \cap \mathbf{R}^L_{--} = \emptyset$.

2. Let $z = -L(\max_{i=1,\ldots,I} ||\omega_i||_\infty, \ldots, \max_{i=1,\ldots,I} ||\omega_i||_\infty)$. Use the Shapley-Folkman Theorem to show that

$$(\text{con } B) \bigcap \left(z + \mathbf{R}^L_{--}\right) \quad = \quad \emptyset. \tag{2}$$

3. Use Minkowski's Theorem to find a price $p \neq 0$ separating B from $z + \mathbf{R}^L_{--}$.

4. Verify that $p \geq 0$ and p satisfies the conclusion of the theorem.

References

[1] Anderson, Robert M., "An Elementary Core Equivalence Theorem," *Econometrica*, 46(1978), 1483-1487.

[2] Anderson, Robert M., "The Core in Perfectly Competitive Economies," Chapter 14 in Robert J. Aumann and Sergiu Hart (editors), *Handbook of Game Theory with Economic Applications*, volume I, 1992, 413-457. Amsterdam: North-Holland Publishing Company

[3] Dierker, Egbert, "Gains and Losses at Core Allocations," *Journal of Mathematical Economics*, 2(1975), 119-128.

[4] Edgeworth, Francis Y. (1881), *Mathematical Psychics*. London: Kegan Paul.

[5] Walras, Leon (1874), *Eléments d'économie politique pure*. Lausanne: L. Corbaz.