Economics 201b
Spring 2010
Problem Set 6 Solution

1. There are many foreign embassies in the Washington, DC. In fact, there is an area of the city where quite a few of them are very close to each other. As you walk along one of the streets you observe three embassies standing next to each other. Each embassy has a flagpole with its national flag flying in the wind. You know that the flagpole height chosen by each embassy depends continuously on the heights chosen by other two embassies. (For instance, having too tall a pole compared to the neighbors would be ostentatious, whereas having one too short would look stingy.) Moreover, having observed heights chosen by the neighbors, each embassy has a single, most favorite height to set. DC ordinance imposes an upper limit of 100 feet on flagpole heights of the embassies. The choices of flagpole heights are in equilibrium when no one wishes to change the height of their flagpole. Prove that there exists an equilibrium.

   Solution. To begin, note how little is known about the optimal height choice by embassies. It might be the case that neither of embassies would like the have the highest or the lowest flagpole, then it is clear that there is an equilibrium when all flagpole heights converge to some average value. However, it is also possible that some embassy would like to set its height as $\frac{\max_i h(i)+100}{2}$, i.e. always to have (one of the) highest flagpole(s). All we know is that embassy choice is a continuous function on a closed and bounded set in $\mathbb{R}^3_+$. But, this information is sufficient to apply a fixed-point argument. Notice that is exactly how we prove the existence of equilibrium in AD economy.

   So, define closed and bounded set $D = [0, 100] \times [0, 100] \times [0, 100] \subset \mathbb{R}^3_+$. Heine-Borrel theorem implies that D is compact in $\mathbb{R}^3_+$, moreover, it is clear that it is convex. Now, define $f : D \to D$ as a function that encodes the optimal choice of each embassy, given the choices of its neighbors. Brouwer's fix point theorem suggest that $\exists_{x^* \in D}\ f(x^*) = x^*$, i.e. there is an equilibrium when no one wishes to change the height of their flagpole.

2. Consider a two-person, two-good exchange economy where all agents have the same utility function, $i = 1, 2$:

$$u(x_{1i}, x_{2i}) = \max\{2\min\{2x_{1i}, x_{2i}\}, \min\{x_{1i}, 4x_{2i}\}\}, \quad \omega_1 = \omega_2 = (1, 1).$$

   (a) Draw the indifference curves for one of the consumer. Are this consumer's preferences convex?

   Solution. The preferences are not convex: the indifference curves have kinks along the lines $x_2 = 2x_1$, $x_2 = \frac{1}{2}x_1$ and $x_2 = \frac{1}{4}x_1$. See Figure 1.

(b) Draw the Edgeworth box for this economy, denoting Pareto set, individually rational and core allocations.

Solution. We know that in Edgeworth box economy, contract curve is equal to the core of the economy, thus, it is a set of all individually rational Pareto optimal allocations. See Figure 2.

(c) Now suppose we have an economy of $I \in \mathbb{N}$ identical consumers with $I \geq 2$, each of which has the same preferences as the consumers described above and endowments are $(\omega_1, \omega_2) = (1 + 3\alpha, 2 - \alpha)$ where $\alpha \in (0, 1)$. Find a necessary and sufficient condition on α that must be satisfied for there to exist a Walrasian equilibrium of this economy. Show that as I increases, the set of $a \in (0, 1)$ that satisfy the condition you found increases in size.

Solution. As with any competitive equilibrium problem, it helps to first characterize each agent's demand as a function of prices. Here, this can be summed up by considering three cases for the price vector:

$$\begin{aligned}
&\text{If } \frac{p_1}{p_2} > \frac{1}{3}, && \text{then } x_2(p_1,p_2) = 2x_1(p_1,p_2). \\
&\text{If } \frac{p_1}{p_2} = \frac{1}{3}, && \text{then } x_2(p_1,p_2) = 2x_1(p_1,p_2) \\
& && \text{or } x_2(p_1,p_2) = \tfrac{1}{4}x_1(p_1,p_2). \\
&\text{If } \frac{p_1}{p_2} < \frac{1}{3}, && \text{then } x_2(p_1,p_2) = \frac{1}{4}2x_1(p_1,p_2).
\end{aligned}$$

All consumers in this economy are identical, so if our equilibrium prices are characterized by $\frac{p_1^*}{p_2^*} > \frac{1}{3}$, then we have $x_2^* = 2x_1^*$, and market clearing implies that we must have $\omega_{2i}^* = 2\omega_{1i}^*$ $\forall i$. However, this is only possible if $\alpha = 0$, and we have constrained $\alpha \in (0, 1)$. Thus, we cannot have $\frac{p_1^*}{p_2^*} > \frac{1}{3}$ in equilibrium.
Similarly, we also cannot have $\frac{p_1^*}{p_2^*} < \frac{1}{3}$ as this would require $\alpha = 1$. Thus, in any equilibrium, we must have $\frac{p_1^*}{p_2^*} = \frac{1}{3}$. In this case, some agents can consume on the $x_2 = 2x_1$ line, while others consume on the $x_2 = \frac{1}{4}x_1$ line (and note that the endowment will lie somewhere between these two lines, depending on the value of α).
Now we are in a position to show when equilibria exist. Denote a "type a" agent as one who consumes on $x_2 = 2x_1$ line and a "type b" agent as one who consumes on $x_2 = \frac{1}{4}x_1$ line, and further let I_a and $I_b \in \mathbb{Z}_+$ denote the number of agents of each type in equilibrium. In equilibrium we must satisfy three equations: budget balancing for the a agents, budget balancing for the b agents, and market clearing for one of the goods (we'll use good 1).
We set $p_1 = 1$ and $p_2 = 3$. The wealth of each type of agent is $W_a = W_b = 1 + 3\alpha + 3(2 - \alpha) = 7$, which is independent of α. We may now write the

budget constraint for the a agents as $x_{1a} + 3x_{2a} = 7$. Applying $x_{2a} = 2x_{1a}$ yields that $x_{1a} = 1$ and $x_{2a} = 2$ in an equilibrium. Similarly, we have $4x_{2b} = x_{1b}$, implying that $x_{1b} = 4$ and $x_{2b} = 1$. Market clearing for good 1 implies that $I_a \cdot x_{1a} + I_b \cdot x_{1b} = (I_a + I_b)(1 + 3\alpha)$ (i.e. the total demand for good 1 equals the total endowment of good 1). Solving this for α yields:

$$\alpha = \frac{I_b}{I}$$

That is, α must be equal to the ratio of type b agents to the total number of agents. What values could this ratio take? For two agents, this ratio could be equal to 0, $\frac{1}{2}$, or 1. However, we cannot have $\alpha = 0$ or $= 1$, so the only option $\alpha = \frac{1}{2}$, which yields an equilibrium with one agent of each type. Similarly, for three total agents, we can have $\alpha = \frac{1}{3}$ or $= \frac{2}{3}$. That is, for $\alpha = \frac{1}{3}$, in equilibrium the agents will sort themselves so that two are of type a while one is of type b, while for $\alpha = \frac{2}{3}$ there will be one agent of type a and two of type b (recall that the agents are indifferent between either type at $\frac{p_1^*}{p_2^*} = \frac{1}{3}$).
Generalizing this, our necessary and sufficient condition on α, given the total number of agents I, is:

$$\alpha \in A = \left\{ \frac{n}{I} : \ n \in \mathbb{N} \text{ and } n < I \right\}$$

Note that as I increases, the number of elements in the set A increases as well: each one-agent increase in I increases the set of possible values for α by one.

(d) Explain in a few sentences how these results relate to Theorem 2 in the Lecture Notes 11. That is, relate your above results to the fact that, in this economy, we can show that $\forall \alpha \in (0,1) \quad \exists p^* \gg 0, \ \ p^* \in \Delta^0$ with $0 \in conE(p^*)$ and $x_i^* \in D_i(p^*)$ such that

$$\frac{1}{I} \sum_{\ell=1}^{L} p_\ell^* \left| \left(\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} \omega_i \right)_l \right| \leq \frac{2L}{I} \max\{\|\omega_i\|_\infty : \ i = 1, ..., I\}$$

Fix $\alpha = \frac{1}{3}$ and compute an explicit bound for the *market value* of the surpluses and shortages in the economy. Verify that the bound provided by the theorem is tight enough.

Solution. We have just showed above that more and more values of α can generate economies with exact Walrasian equilibria, when we add more agents to this economy (i.e., when we increase I). Thus, when we take any $\alpha \in (0,1)$, the greater the number of agents, the more likely it is that the we pick will be "close" to one of the the values of that generates an exact equilibrium. This means that the excess demands given α will not be too large for large I, which is what is predicted by Theorem 2.

Lets compute the explicit bound for the *market value* of the surpluses and shortages in the economy and verify that the bound provided by the theorem is tight enough. In our case, there are two type of agents, I_a and I_b, with identical endowments. Consequently, we need to show that $\forall\alpha \in (0,1) \quad \exists p^* \gg 0, \;\; p^* \in \Delta^0$ with $0 \in conE(p^*)$ and $x_i^* \in D_i(p^*)$ such that

$$p_1^* \cdot |I_a \cdot x_{1a}^* + (I - I_a) \cdot x_{1b}^* - I \cdot (1 + 3\alpha)| +$$
$$+ p_2^* \cdot |I_a \cdot x_{2a}^* + (I - I_a) \cdot x_{2b}^* - I \cdot (2 - \alpha)| \leq 4 \max\{1 + 3\alpha, 2 - \alpha\}$$

So, pick $p^* = \left(\frac{1}{4}, \frac{3}{4}\right) \in \Delta^0$. a-type agents demand is given by $x_a^* = (2,1)$ and b-type agents demand is given by $x_b^* = (1,4)$. We have

$$\frac{1}{4}\,|I_a + 4(I - I_a) - I(1 + 3\alpha)| +$$
$$+ \frac{3}{4}\,|2I_a + (I - I_a) - I(2 - \alpha)| \leq 4 \max\{1 + 3\alpha, 2 - \alpha\}$$

which simplifies to $|(1-\alpha)I - I_a| \leq \frac{8}{3}\max\{1+3\alpha, 2-\alpha\}$. Now, set $\alpha = \frac{1}{3}$. We have

$$\left|\frac{2}{3}I - I_a\right| \leq \frac{8}{3}\max\{2, 1\frac{2}{3}\} \iff |2I - 3I_a| \leq 16$$

Consider 3 following cases:

$I = 0 \mod 3$. In this just pick $I_a = \frac{2}{3}I,\; 1 \leq I_a \leq I - 1$. We have $|2I - 3I_a| = 0 \leq 16$.

$I = 1 \mod 3$. In this case $I = h + 1$, where $h = 0 \mod 3$ and $h \geq 3$. Again, pick $I_a = \frac{2}{3}h,\; 1 \leq I_a \leq I - 1$. We have $|2I - 3I_a| = |2h + 2 - 2h| = 2 \leq 16$.

$I = 2 \mod 3$. Consider two possibilities, either $I = 2$, then $I_a = 1$ and $|2I - 3I_a| = 1 \leq 16$, or $I \geq 5$ and $I = h + 2$, where $h = 0 \mod 3$ and $h \geq 3$. Again, pick $I_a = \frac{2}{3}h,\; 1 \leq I_a \leq I - 1$. We have $|2I - 3I_a| = |2h + 4 - 2h| = 4 \leq 16$.

So, we have verified that the bound provided by the theorem is tight enough.

3. Consider four-person, two-good pure exchange economy where agents have endowments $\omega_1 = \omega_2 = (10, 10)$ and $\omega_3 = \omega_4 = (10, 30)$ and the same utility function $U_i(x_{1i}, x_{2i}) = \log x_{1i} + \log x_{2i},\; i = 1, 2, \ldots, 4$. For each allocation vector given below show whether the it is Pareto optimal; is in the core (if not, provide a blocking coalition); can be supported as a competitive equilibria for some price vector. Explain your reasoning.

   (a) $x_1 = x_2 = (7.5, 15)$ and $x_3 = x_4 = (12.5, 25)$.

Solution. The allocation $x_1 = x_2 = (7.5, 15)$ and $x_3 = x_4 = (12.5, 25)$ is Pareto optimal, is in the core and can be supported as a competitive equilibria for some price vector $\frac{p_1^*}{p_2^*} = 2$. To prove this claim, we will first show that this allocation is competitive equilibrium, thus, Pareto optimality follows from the First Welfare Theorem. Since any Walrasian equilibrium lies in the core by the Strong First Welfare Theorem, it has to be core allocation as well.

Notice that $\frac{\partial U_i}{\partial x_{1i}}|_{x_{1i}=0^+} = \frac{\partial U_i}{\partial x_{2i}}|_{x_{2i}=0^+} = +\infty$, so consumer i will never demand zero amount of any commodity and the prices have to be strictly positive. Observe $MRS_i = \frac{\partial U_i \backslash \partial x_{1i}}{\partial U_i \backslash \partial x_{2i}} = \frac{x_{2i}}{x_{1i}} = \frac{p_1}{p_2} = \frac{1}{p}$ for all agents. Also, since

$$\frac{80}{40} = \frac{2\omega_{21} + 2\omega_{23}}{2\omega_{11} + 2\omega_{13}} = \frac{2x_{21} + 2x_{23}}{2x_{11} + 2x_{13}} = \frac{1}{p} \implies p^* = \frac{1}{2}.$$

Thus, an allocation $x_1 = x_2 = (7.5, 15)$ and $x_3 = x_4 = (12.5, 25)$ is indeed a Walrasian equilibrium with $p^* = \frac{1}{2}$.

(b) $x_1 = x_2 = (\sqrt{50}, 2\sqrt{50})$ and $x_3 = x_4 = (20 - \sqrt{50}, 40 - 2\sqrt{50})$.

Solution. The allocation $x_1 = x_2 = (\sqrt{50}, 2\sqrt{50})$ and $x_3 = x_4 = (20 - \sqrt{50}, 40 - 2\sqrt{50})$ is a Pareto optimal, but it is not in the core, and, consequently, it cannot be Walrasian equilibrium. To see this, observe that the proposed allocation is a Pareto optimal one, since it is interior and MRS for all agents are equalized (since the utility functions are strongly monotone, any Pareto optimal allocation has to be interior).

Now, notice that agents 1 and 2 receive the same utility in the proposed allocation as they would by just consuming their endowment. As it was shown in the lecture, such allocation cannot be in the core of 2−fold replica economy, and, consequently, it cannot be supposed as competitive equilibrium. Similarly to what was done in lecture, consider a blocking coalition of agents $1, 2$ and 3. It is easy to see that this coalition blocks the current allocation by dividing their aggregate endowment, for instance, in the following way: $x_1 = x_2 = (8, 13)$ and $x_3 = (14, 24)$.

(c) $x_1 = (8, 12), x_2 = (9, 11), x_3 = (12, 23)$ and $x_4 = (11, 29)$.

Solution. The allocation $x_1 = (8, 12), x_2 = (9, 11), x_3 = (12, 23)$ and $x_4 = (11, 29)$ is neither Pareto optimal nor is in the core. It cannot be Pareto optimal because MRS are not equalized among agents, and (weak) Pareto optimality of core allocations implies that it cannot be in the core.

Since all assumptions of the First Welfare Theorem are satisfied in this economy, the proposed allocation cannot be sustained as a Walrasian equilibrium because otherwise it has be Pareto optimal.

4. Give an example of a three-person, two-good pure exchange economy where all agents have the same utility function $U_i(x_{1i}, x_{2i}) = \log x_{1i} + \log x_{2i},$. Find a set of

integer endowments for these agents along with a Pareto optimal, individually rational *integer* allocation that is not in the core.

Solution. Note that in any two-agent economy core is just a contract curve, i.e. set of all individually rational Pareto optimal allocations. So, to find a counterexample to this fact, we need to consider three agents. Consider the endowments $\omega_1 = (1, 4)$, $\omega_2 = (4, 1)$, $\omega_3 = (1, 1)$ then $x_1 = x_2 = x_3 = (2, 2)$ is a Pareto optimal, individually rational allocation that is not in the core. Notice that the coalition of agents 1 and 2 can get together and split their goods down the middle so that each agent receives the preferred allocation $(2.5, 2.5)$. But, it is clearly an individual rational one. So, we need only to verify Pareto optimality, and this is straightforward as well — since the solution is in the interior, and utilities are smooth, it suffices to check that the marginal rates of substitution are equal. Since $MRS_i(x_{1i}, x_{2i}) = \frac{x_{2i}}{x_{1i}}$ for all agents and is 1 under the current allocation, we have Pareto optimality.

5. Consider a pure exchange economy with $H = 2$ consumers and L goods, with social endowment $\bar{\omega} \in \mathbb{R}^L_{++}$. In this question, we will consider the n-fold replica of this economy. In the n-fold replica, there are $2n$ agents, of whom n (referred to as type 1 agents) have preferences and endowments identical to those of agent 1 in the original economy, and n (referred to as type 2 agents) have preferences and endowments identical to those of agent 2 in the original economy.

   (a) Let p^* be an equilibrium price vector for the original economy. Show that p^* is also an equilibrium price vector for the (larger) n-fold replica economy.

   Solution. Let p^* be an equilibrium price vector for the original economy. Then it must be the case that $x_1^* \in D(p^*, \omega_1)$, $x_2^* \in D(p^*, \omega_2)$ and markets clear $x_1^* + x_2^* = \omega_1 + \omega_2$. Now, consider $n-$fold replica economy. Since all type 1 and 2 agents have the same preference and endowments as in the original economy, their demand sets are unchanged when they face price p^* and we can show that markets clear in $n-$fold replica economy. Indeed,

$$\sum_{i=1}^{2n} x_i^*(p^*, \omega_i) = \sum_{i=1}^{n} x_i^*(p^*, \omega_i) + \sum_{i=n+1}^{2n} x_i^*(p^*, \omega_i) =$$
$$= n\left(x_1^*(p^*, \omega_1) + x_2^*(p^*, \omega_2)\right) = n(\omega_1 + \omega_2) =$$
$$= \sum_{i=1}^{n} \omega_i + \sum_{i=n+1}^{2n} \omega_i = \sum_{i=1}^{2n} \omega_i$$

   Thus, p^* is an equilibrium price vector in n-fold replica.

   (b) Now, lets consider a special case where there are two commodities and two type of agents. Type 1 is characterized as

$$U_1(x_{11}, x_{21}) = x_{11}x_{21}, \quad \omega_1 = (10, 0)$$

and type 2 is characterized as

$$U_2(x_{12}, x_{22}) = (x_{12})^{\frac{1}{2}}(x_{22})^{\frac{1}{2}}, \;\; \omega_2 = (0, 10).$$

Show that the allocation $x_1 = x_2 = (5, 5)$, (x_1 to agents of type 1 and x_2 to agents of type 2) is in the core for all levels of replication n.

Solution. We will prove the claim by showing that the allocation $x_1 = x_2 = (5, 5)$, is a (unique) competitive equilibrium of the original economy and then appeal to the Strong First Welfare Theorem to argue that it must be in the core for any level of replication n. Proving that $x_1 = x_2 = (5, 5)$ is unique equilibrium is actually quite easy. We have symmetric Cobb-Douglas consumers, thus, everything is smooth, strictly quasi-concave and aggregate demand curves slope downwards. Consequently, we have a unique symmetric Walrasian equilibrium with $p_1^* = p_2^*$ and $x_1^* = x_2^* = (5, 5)$.

(c) Continue to assume two-good, two-agent type economy given above. Show that the allocation $x_1 = (9, 9), \;\; x_2 = (1, 1)$, is in the core for the original economy with one agent of the each type and is *not* in the core for the n-fold replica with $n \geq 2$. Discuss.

Solution. Notice that for the reason just given in (b) the set of all Pareto optimal allocations is just a diagonal of the Edgeworth box connecting two origins, O_1 and O_2. Thus, an allocation $x_1 = (9, 9), \;\; x_2 = (1, 1)$ is in the core because it satisfies individual rationality constraints, i.e. $U_1(9, 9) > U_1(10, 0)$ and $U_2(9, 9) > U_2(0, 10)$.
To show that the allocation $x_1 = (9, 9), \;\; x_2 = (1, 1)$ is not in the core for the n-fold replica with $n \geq 2$, we will show that it is not in the core of 2-fold replica. Because any coalition that is active in n-fold replica is also active in (n+1)-replica, we obtain the result we seek. So, notice that current allocation achieves relatively low level of utility for type 2 agents. As it has been shown in the lecture, together with one type 1 agent they can block the allocation. For instance they block it with an allocation $x_1 = (6, 16)$ and $x_3 = x_4 = (2, 2)$.

6. Give an example of acyclic preference relation that is *not* transitive.

Solution. Note that any transitive preference relation is acyclic, this follows directly from definition of acyclicity. While we usually work with transitive preferences, sometimes it might be too strong of an assumption. As an example of acyclic preference relation that is not transitive consider following preference relation: an individual prefers apples (A) to bananas (B) and bananas to cherries (C). However, one is indifferent between apples and cherries, or formally:

$$A \succ B \succ C \text{ and } A \sim C$$

This relation is clearly acyclic because there is no cycle (notice that would not be true if $C \succ A$). But this preference relation is not quasi-transitive, and, thus, is not transitive, since quasi-transitivity would require $A \succ C$.

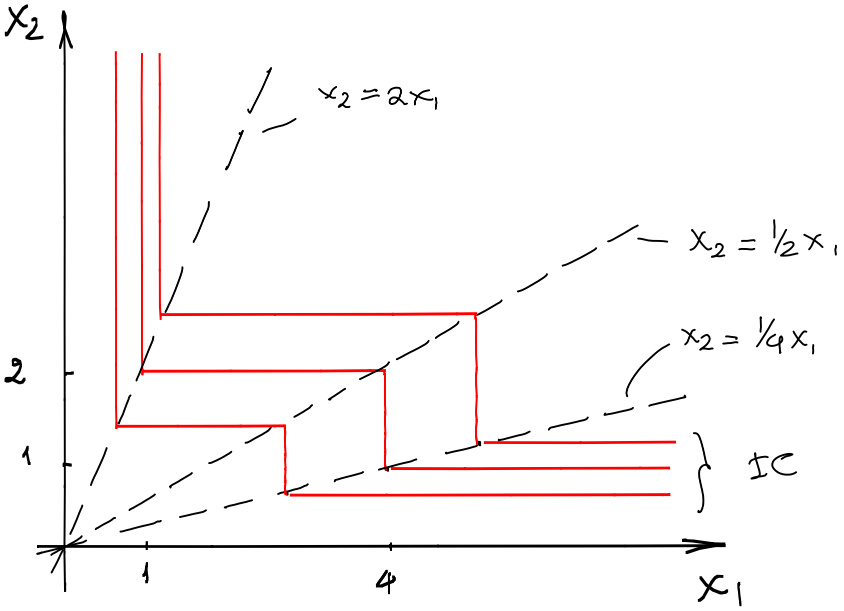

Figure 1. Indifference curves are not convex.

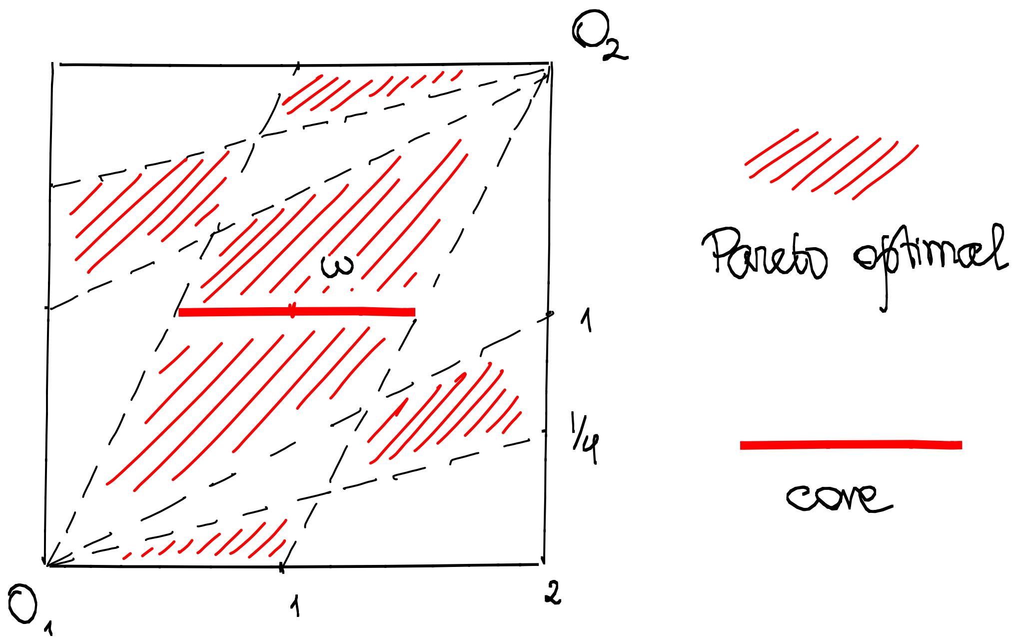

Figure 2. Edgeworth Box