Economics 201B–Second Half

Lecture 7, 4/6/10

Existence of Walrasian Equilibrium

Review of Upperhemicontinuity in $\mathbf{R}^n$

Definition 1 Let $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$. Suppose $\Psi : X \to Y$ is a correspondence. Ψ is *upper hemicontinuous (uhc)* at $x_0 \in X$ if, for every open set $V \supseteq \Psi(x_0)$, there is an open set U with $x_0 \in U$ such that

$$\Psi(x) \subseteq V \text{ for every } x \in U \cap X$$

This says Ψ doesn't "implode in the limit" at x_0

Definition 2 Suppose $X \subseteq \mathbf{R}^m$, $Y \subseteq \mathbf{R}^n$. A correspondence $\Psi : X \to Y$ is called *closed-valued* if $\Psi(x)$ is a closed subset of $\mathbf{R}^n$ for all x; Ψ is called *compact-valued* if $\Psi(x)$ is compact for all x.

Theorem 3 (Not in de la Fuente) *Suppose $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^m$, and $\Psi : X \to Y$ is a correspndence.*

- *If Ψ is closed-valued and uhc, then Ψ has closed graph.*
- *If Y is compact and Ψ has closed graph, then Ψ is uhc.*

Theorem 4 (Brouwer's Fixed Point Theorem) *Suppose*

$A \subset \mathbf{R}^L$ is nonempty, convex, compact, and $f : A \to A$ is continuous. Then f has a fixed point, i.e.

$$\exists_{x^* \in A}\ f(x^*) = x^*$$

Review material from Lecture 13 in Econ 204. Recall that the Scarf Algorithm gives a constructive way to find approximate fixed points: given any $\varepsilon > 0$, the algorithm will eventually find a point x^*_ε such that

$$|f(x^*_\varepsilon) - x^*_\varepsilon| < \varepsilon$$

Theorem 5 (Kakutani's Fixed Point Theorem) *Suppose $A \subset \mathbf{R}^L$ is nonempty, compact, convex, and $f : A \to A$ is a correspondence (recall $f(a) \in 2^A$) such that*

1. *f is nonempty-valued: $\forall_{a \in A}\ f(a) \neq \emptyset$.*

2. *f is convex-valued: $\forall_{a \in A}\ f(a)$ is convex.*

3. *f is closed-valued.*

4. *f is upper hemicontinuous.*

Then f has a fixed point, i.e.

$$\exists_{x^* \in A}\ x^* \in f(x^*)$$

Now, we turn to existence: Let

$$\begin{aligned} \Delta &= \left\{ p \in \mathbf{R}^L_+ : \sum_{\ell=1}^{L} p_\ell = 1 \right\} \\ \Delta^0 &= \left\{ p \in \mathbf{R}^L_{++} : \sum_{\ell=1}^{L} p_\ell = 1 \right\} \end{aligned}$$

Proposition 6 (17.C.1) Debreu-Gale-Kuhn Nikaido Lemma *Suppose $z : \Delta^0 \to \mathbf{R}^L$ is a function satisfying*

1. *continuity*

2. *Walras' Law*

$$\forall_{p \in \Delta^0}\ p \cdot z(p) = 0$$

3. *bounded below:*

$$\exists_{x \in \mathbf{R}^L} \forall_{p \in \Delta^0}\ z(p) \geq x$$

4. Boundary Condition: If $p_n \to p$ where $p \in \Delta \setminus \Delta^0$, then

$$|z(p_n)| \to \infty$$

Then there exists $p^* \in \Delta^0$ such that

$$z(p^*) = 0$$

Remark: A common misperception is that the boundary condition says that if the price of a good goes to zero, then excess demand for *that* good goes to infinity. If the price of good ℓ goes to zero and the prices of the other goods are all bounded away from zero, then the demand for good ℓ goes to infinity. However, if the prices of two goods are going to zero, it is entirely possible for the demand for one of them to stay bounded, or for there to be two subsequences such that the demand for one is bounded on the first subsequence and the demand for the second is bounded on the second subsequence.

Outline of proof:

- Define a correspondence $f : \Delta^0 \to \Delta$ (so $f(p) \in 2^{\Delta}$) by

  $$f(p) = \{q \in \Delta : q \cdot z(p) \geq q' \cdot z(p) \text{ for all } q' \in \Delta\}$$

  f identifies the goods in highest excess demand.

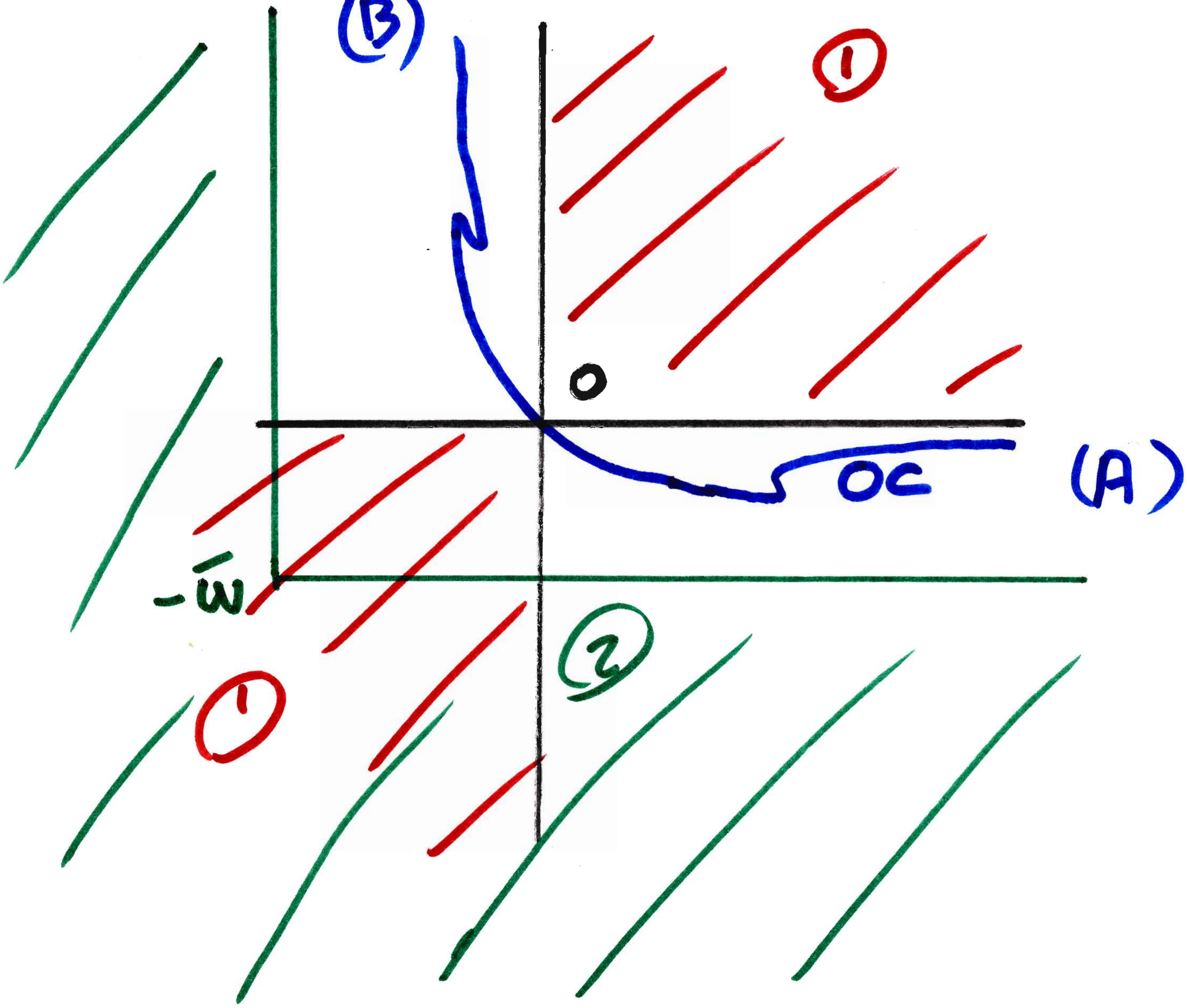
- Since Δ^0 is not compact, we extend f to Δ which is compact in such a way that f is upper hemi-continuous.

- Verify that if $p^* \in f(p^*)$, then $p^* \in \Delta^0$ and $z(p^*) = 0$.

- Check that f satisfies the hypotheses of Kakutani's Theorem.

- By Kakutani's Theorem, there exists $p^* \in \Delta$ such that $p^* \in f(p^*)$, so $p^* \in \Delta^0$ and $z(p^*) = 0$.

Will prove this in detail on Thursday.



(B)
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Corollary 7 *In a pure exchange economy (recall, $X_i = \mathbf{R}^L_+$, $\bar{\omega} \gg 0$, preferences complete, transitive, locally nonsatiated) in which preferences are continuous, strictly convex and strongly monotone, there is a Walrasian Equilibrium.*

Proof: Define $z : \Delta^0 \to \mathbf{R}^L$ by

$$z(p) = \left(\sum_{i=1}^{I} D_i(p) \right) - \bar{\omega}$$

We need to show that z satisfies the hypotheses of Debreu-Gale-Kuhn-Nikaido.

- z is a continuous function: 201A
- Walras' Law: holds with equality because of strong monotonicity, although we've also seen that local nonsatiation is enough.
- Bounded below: $D_i(p) \geq 0$, so $z(p) \geq -\bar{\omega}$.
- Boundary Condition: We must show the following: if $p^n \in \Delta^0$ and $p^n \to p$, where $p \in \Delta \backslash \Delta^0$, then $|z(p^n)| \to \infty$. We have $\bar{\omega} = \sum_i \omega_i >> 0$, so $p \cdot \sum_i \omega_i > 0$, so $p \cdot \omega_i > 0$ for some i, WLOG $i = 1$. We claim that $|D_1(p^n)| \to \infty$.
  - If not, we can find a subsequence p^{n_k} s.t. $D_1(p^{n_k})$ is bounded; by the Bolzano-Weierstrass Theorem, there is a further subsequence (still denoted p^{n_k}) such that $D_1(p^{n_k}) \to x$ for some $x \in \mathbf{R}^L$.
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$$\begin{aligned} p \cdot x &= \lim_{k\to\infty} p^{n_k} \cdot D_1(p^{n_k}) \\ &= \lim_{k\to\infty} p^{n_k} \cdot \omega_1 \text{ (Walras' Law with Equality)} \end{aligned}$$

$$
\begin{aligned}
&= p \cdot \omega_1 \\
&> 0
\end{aligned}
$$

– Without loss of generality, assume $p_1 = 0$. Since $p \cdot x > 0$, there is a good ℓ (WLOG $\ell = 2$) such that $p_\ell > 0$ and $x_\ell > 0$. By strong monotonicity, $x + (1, 0, ..., 0) \succ_1 x$. Further, by the continuity of $\succeq_1$, there exists $\varepsilon > 0$ such that

$$
\begin{aligned}
x + (1, -\varepsilon, ..., 0) &\geq 0 \\
x + (1, -\varepsilon, ..., 0) &\succ_1 x
\end{aligned}
$$

Let $y = x + (1, -\varepsilon, ..., 0)$. Since $D_1(p^{n_k}) \to x$, there exists K_1 such that $y \succ_1 D_1(p^{n_k})$ for every $k > K_1$. Note that

$$
\begin{aligned}
\lim_{k\to\infty} p^{n_k} \cdot y &= p \cdot y \\
&= p \cdot (x + (1, -\varepsilon, ..., 0)) \\
&= p \cdot x - p_2 \varepsilon \\
&< p \cdot x \\
&= p \cdot \omega_1 \\
&= \lim_{k\to\infty} p^{n_k} \cdot \omega_1
\end{aligned}
$$

Thus, there exists K_2 such that $p^{n_k} \cdot y < p^{n_k} \cdot \omega_1$ for every $k > K_2$. Thus, for every $k > \max\{K_1, K_2\}$, we have $y \succ_1 D_1(p^{n_k})$ and $p^{n_k} \cdot y < p^{n_k} \cdot \omega_1$, a contradiction that shows that $|D_1(p^n)| \to \infty$.

Since

$$
\begin{aligned}
\sum_{i=1}^{I} D_i(p^n) &\geq D_1(p^n) \geq 0 \\
\text{so } \left| \sum_{i=1}^{I} D_i(p^n) \right| &\geq |D_1(p^n)| \to \infty
\end{aligned}
$$

$$\begin{aligned} \text{so } |z(p^n)| &= \left|\sum_{i=1}^{I} D_i(p^n) - \bar{\omega}\right| \\ &\geq \left|\sum_{i=1}^{I} D_i(p^n)\right| - |\bar{\omega}| \\ &\to \infty \end{aligned}$$

which proves the Boundary Condition.

Since z satisfies the hypotheses of D-G-K-N, there exists $p^* \in \Delta^0$ such that $z(p^*) = 0$. Let $x_i^* = D_i(p^*)$. Then (p^*, x^*) is a Walrasian Equilibrium. ∎