Economics 201B–Second Half

Lecture 11, 4/20/10

17.I and Handout: Nonconvex Preferences and Indivisibilities

- Why do we care?
  - General story that justifies convexity (diminishing MRS along each indifference curve) works for "most" goods, but if there is a single pair of goods for which preferences are nonconvex, or if there is a single good which is indivisible, Existence of Walrasian Equilibrium and the Second Welfare Theorem fail.
    * $\frac{1}{2}$ house in Berkeley and $\frac{1}{2}$ house in SF
    * Two trips to Winnemucca are not preferable to one trip to Salt Lake if you like to ski.
    * Painting a room with orange and green stripes is not preferable to solid orange or solid green.

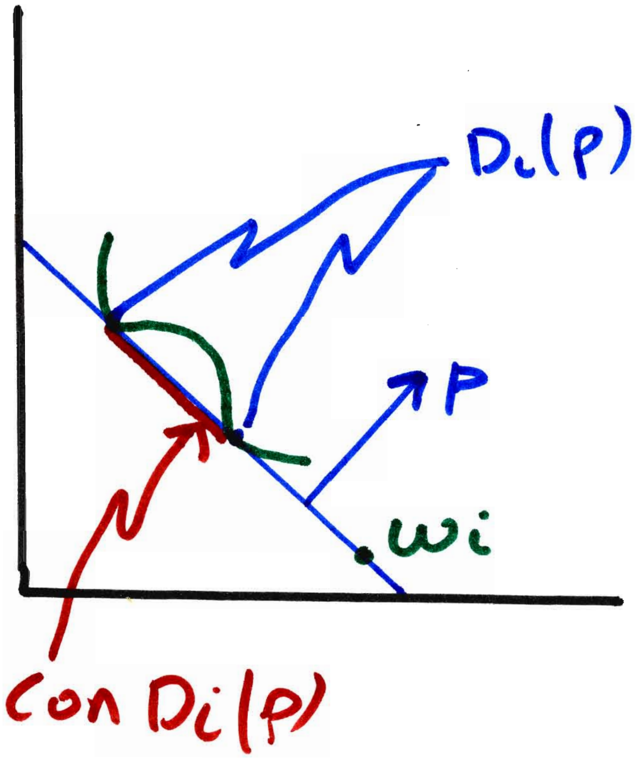
- **Theorem 1 (Shapley-Folkman)** *Suppose $x \in \text{con}\,(A_1 + \cdots + A_I)$, where $A_i \subset \mathbf{R}^L$. Then we may write $x = a_1 + \cdots + a_I$, where $a_i \in \text{con}\,A_i$ for all i and $a_i \in A_i$ for all but L values of i.*

  Proof: The proof is in the handout, it just uses the fact that $m \geq L + 1$ vectors in $\mathbf{R}^L$ must be linearly dependent.■

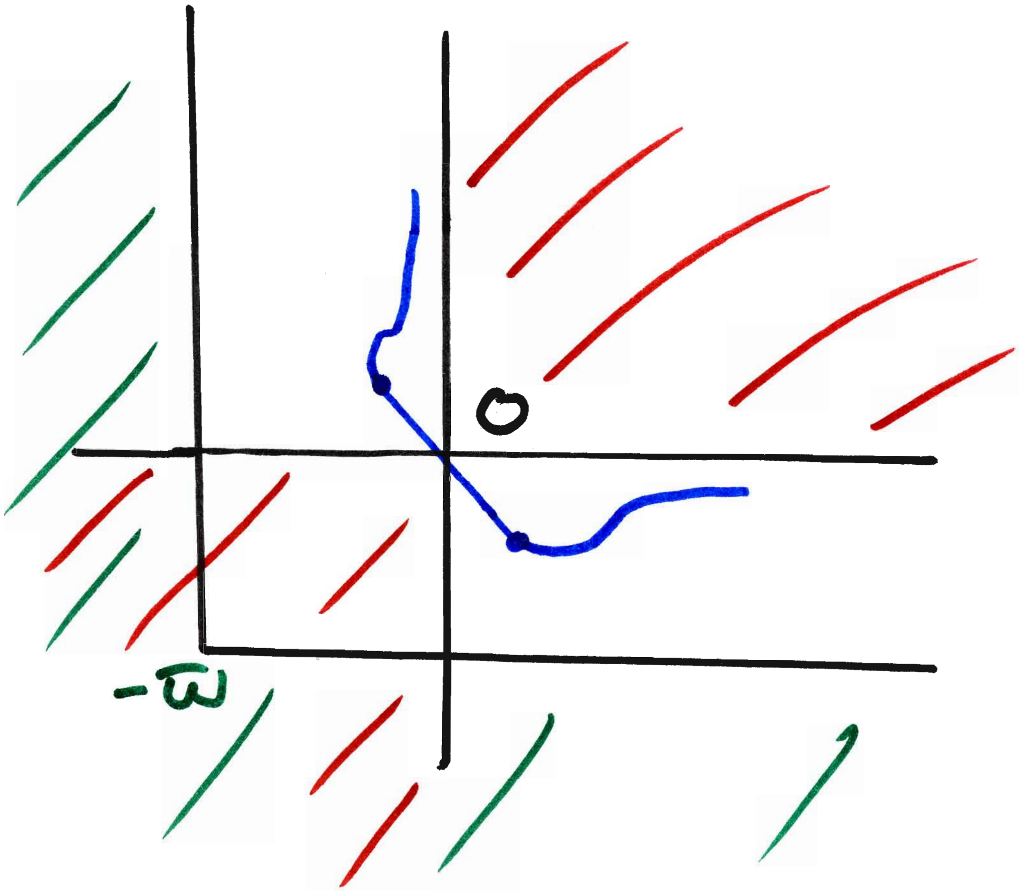
  Theorem 2 *Suppose we are given a pure exchange economy, where for each $i = 1, \ldots, I$, $\succ_i$ satisfies*

  1. *continuity: $\{(x, y) \in \mathbf{R}^L_+ \times \mathbf{R}^L_+ : x \succ_i y\}$ is relatively open in $\mathbf{R}^L_+ \times \mathbf{R}^L_+$;*

Approximate Equilibrium with Nonconvexities



"offer curve"



2. *for each individual i, the consumption set is $\mathbf{R}^L_+$, i.e. each good is perfectly divisible, and each agent is capable of surviving on zero consumption;*

3. *acyclicity: there is no collection $x_1, x_2, \ldots, x_m$ such that $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m \succ_i x_1$;*

4. *strong monotonicity:*

$$x > y \Rightarrow x \succ_i y$$

Then there exists $p^* \in \Delta^0$ with $0 \in \text{con } E(p^*)$ and $x_i^* \in D_i(p^*)$ such that

$$\frac{1}{I}\sum_{\ell=1}^{L} p_\ell^* \left| \left(\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} \omega_i \right)_\ell \right| \leq \frac{2L}{I} \max\{\|\omega_i\|_\infty : i = 1, \ldots, I\}$$

where $\|x\|_\infty = \max\{|x_1|, \ldots, |x_L|\}$.

The inequality bounds the *market value* of the surpluses and shortages in the economy. There might be a large surplus of a good whose price is nearly zero; although the statement seems to allow a large shortage of a good whose price is nearly zero, in practice goods which are in short supply tend not to be inexpensive. Dividing by I on the left side of the inequality expresses the market value of the surplus and shortages in per capita terms. On the right side, we typically assume that the number of individuals (I) is much larger than the number of goods (L); this will certainly be true if we consider a model in which goods are somewhat aggregated (food, clothing, housing, transportation, $\ldots$,; or hamburger, steak, milk, ...); it will be false if we consider each dwelling unit to be an entirely separate commodity from every other dwelling unit. Remember that we are using an exchange economy to study the allocation of consumption, taking the production decisions as exogenous; thus, $\max\{\|\omega_i\|\}$ should be thought of as the maximum resources devoted to *consumption* by any individual, which is typically much less than that individual's total wealth.

The fact that $0 \in \text{con } E(p^*)$ has its own, separate interpretation. Imagine that not everyone trades at exactly the same time, but people come to the market at different times, and choose consumption vectors out of their demand sets. If there is a little inventory in the market, then the demands can be accommodated for a while. Once the inventory starts running out of some goods, the price can be changed very slightly to shift the demand. Very small shifts in price can move the demand around $E(p^*)$; by spending various amounts of time at different points in $E(p^*)$, the market can effectively produce any excess demand in con $E(p^*)$, in particular zero excess demand.

Theorem 3 *Suppose we are given a pure exchange economy, where for each $i = 1, \ldots, I$, $\succ_i$ satisfies*

1. *continuity: $\{(x, y) \in \mathbf{R}_+^L \times \mathbf{R}_+^L : x \succ_i y\}$ is relatively open in $\mathbf{R}_+^L \times \mathbf{R}_+^L$;*
2. *for each individual i, the consumption set is $\mathbf{R}_+^L$, i.e. each good is perfectly divisible, and each agent is capable of surviving on zero consumption;*
3. *acyclicity: there is no collection $x_1, x_2, \ldots, x_m$ such that $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m \succ_i x_1$;*

Then there exists $p^* \gg 0$ and $x_i^* \in D_i(p^*)$ such that

$$\begin{aligned} & \frac{1}{I}\sum_{\ell=1}^{L} \max\left\{\left(\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I}\omega_i\right)_\ell, \ 0\right\} \\ \leq & \ 2\sqrt{\frac{L}{I}}\max\{\|\omega_i\|_1 : i = 1, \ldots, I\} \end{aligned} \tag{1}$$

where $\|x\|_1 = \sum_{\ell=1}^{L} |x_\ell|$.

Notice that the theorem does not assume any monotonicity, or even local nonsatiation. It bounds only the shortages in the economy; there may be large surpluses of some goods.

Proof Outline:

– Let

$$\Delta' = \left\{p \in \mathbf{R}^L : \sqrt{\frac{L}{I}} \leq p_\ell \leq 1 \ (\ell = 1, \ldots, L)\right\}$$

Why?

* Convenient to normalize prices by $\|p\|_\infty = 1$, would have to prove Kakutani's Theorem holds on that set, which is not convex. Δ' is convex and compact.
* In the proof of Debreu-Gale-Kuhn-Nikaido, we define the correspondence on Δ^0 and extend it to Δ. Get a fixed point $\hat{p}^* \in \Delta^0$, but we get no control over the price of the cheapest good. This gives us no control over the diameter of the budget set, which bounds the size of the nonconvexity in the demand set.
* By trimming Δ' so all prices bounded below by $\sqrt{L/I}$, we give up the fact that the Kakutani fixed point is in the interior of Δ', but we get control over the diameter of the budget set.

- Consider the correspondence $z : \Delta' \to \mathbf{R}^L$ defined by

$$z(p) = \left(\sum_{i=1}^{I} \text{con } D_i(p) \right) - \bar{\omega}$$

- Acyclicity implies that $z(p) \neq \emptyset$.

- Look at the "Offer Curve"

$$\{x : \exists_{p \in \Delta'}\ x \in z(p)\}$$

  The picture is almost the same as the picture in the convex case.

- Choose a compact set $X \subset \mathbf{R}^L$ such that

$$p \in \Delta' \Rightarrow z(p) \subseteq X$$

  Define a correspondence $f : \Delta' \times X \to \Delta' \times X$ by

$$f(p, x) = \{(q, y) : y \in z(p),\ \forall_{q' \in \Delta'}\ q \cdot x \geq q' \cdot x\}$$

  By Kakutani's Theorem, there exists a fixed point $(p^*, \bar{x}^*)$

$$\begin{aligned} \bar{x}^* &= \sum_{i=1}^{I} \bar{x}_i^* \\ \bar{x}_i^* &\in \text{con } E_i(p^*)\ (i = 1, \ldots, I) \quad (2) \\ \forall_{q \in \Delta'}\ q \cdot \bar{x}^* &\leq p \cdot \bar{x}^* = 0 \quad (3) \end{aligned}$$

- In the proof of Debreu-Gale-Kuhn-Nikaido, we showed that

$$\forall_{q \in \Delta}\ q \cdot \bar{x}^* \leq 0 \Rightarrow \bar{x}^* \leq 0$$

  Use a similar argument and Equation (3) to put an upper bound on the positive components of $\bar{x}^*$.

- From Equation (2) and the Shapley-Folkman Theorem, we can assume that

$$\begin{aligned} \bar{x}_{i_\ell}^* &\in \text{con } E_{i_\ell}(p^*)\ (\ell = 1, \ldots, L) \\ \bar{x}_i^* &\in E_i(p^*) \text{ for } i \notin \{i_1, \ldots, i_L\} \end{aligned}$$

- Choose arbitrarily

$$x^*_{i_1} \in E_{i_1}(p^*), \ldots, x^*_{i_L} \in E_{i_L}(p^*)$$

and let

$$x^*_i = \bar{x}^*_i \text{ for } i \notin \{i_1, \ldots, i_L\}$$

so

$$x^*_i \in E_i(p^*) \text{ for all } i$$

-

$$\sum_{i=1}^{I} x^*_i = \bar{x}^* + \sum_{\ell=1}^{L} \left(x^*_{i_\ell} - \bar{x}^*_{i_\ell}\right)$$

Error Term

- The diameters of the budget sets are bounded above by the endowments and the lower bound on prices in Δ', which bounds the Error Term.

Indivisibilities:

Theorem 3 applies verbatim to the case of indivisibilites, except that one must substitute Q_i for D_i. With indivisibilities, Q_i has closed graph but D_i generally does not.