Economics 101A
(Lecture 4)

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Outline

1. Convexity and concavity II
2. Constrained Maximization
3. Envelope Theorem II
4. Preferences

1 Convexity and concavity

- Alternative characterization of convexity.

- A function f, twice differentiable, is concave if and only if **for all** x the subdeterminants $|H_i|$ of the Hessian matrix have the property $|H_1| \leq 0, |H_2| \geq 0, |H_3| \leq 0$, and so on.

- For the univariate case, this reduces to $f'' \leq 0$ for all x

- For the bivariate case, this reduces to $f''_{x,x} \leq 0$ and $f''_{x,x} * f''_{y,y} - \left(f''_{x,y}\right)^2 \geq 0$

- A twice-differentiable function is strictly concave if the same property holds with strict inequalities.

- Examples.

  1. For which values of a, b, and c is $f(x) = ax^3 + bx^2 + cx + d$ is the function concave over R? Strictly concave? Convex?

  2. Is $f(x, y) = -x^2 - y^2$ concave?

- For Example 2, compute the Hessian matrix

  - $f'_x =$ $\qquad\qquad\qquad$ $, f'_y =$
  - $f''_{x,x} =$ $\qquad\qquad\qquad$ $, f''_{x,y} =$
  - $f''_{y,x} =$ $\qquad\qquad\qquad$ $, f''_{y,y} =$
  - Hessian matrix H :

    $$H = \begin{pmatrix} f''_{x,x} = & & f''_{x,y} = & \\ f''_{y,x} = & & f''_{y,y} = & \end{pmatrix}$$

- Compute $|H_1| = f''_{x,x}$ and $|H_2| = f''_{x,x} * f''_{y,y} - \left(f''_{x,y}\right)^2$

- Why are convexity and concavity important?

- Theorem. Consider a twice-differentiable concave (convex) function over $C \subset R^n$. If the point $\mathbf{x}_0$ satisfies the fist order conditions, it is a global maximum (minimum).

- For the proof, we need to check that the second-order conditions are satisfied.

- These conditions are satisfied by definition of concavity!

- (We have only proved that it is a local maximum)

2 Constrained Maximization

- Ch. 2, pp. 36-42 (38–44, 9th Ed)

- So far unconstrained maximization on R (or open subsets)

- What if there are constraints to be satisfied?

- Example 1: $\max_{x,y} x * y$ subject to $3x + y = 5$

- Substitute it in: $\max_{x,y} x * (5 - 3x)$

- Solution: $x^* =$

- Example 2: $\max_{x,y} xy$ subject to $x \exp(y) + y \exp(x) = 5$

- Solution: ?

- Graphical intuition on general solution.

- Example 3: $\max_{x,y} f(x,y) = x * y$ s.t. $h(x,y) = x^2 + y^2 - 1 = 0$

- Draw $0 = h(x,y) = x^2 + y^2 - 1$.

- Draw $x * y = K$ with $K > 0$. Vary K

- Where is optimum?

- Where dy/dx along curve $xy = K$ equals dy/dx along curve $x^2 + y^2 - 1 = 0$

- Write down these slopes.

Idea: Use implicit function theorem.

- Heuristic solution of system

$$\begin{aligned} &\max_{x,y} f(x,y) \\ &\text{s.t. } h(x,y) = 0 \end{aligned}$$

- Assume:
  - continuity and differentiability of h
  - $h'_y \neq 0$ (or $h'_x \neq 0$)

- Implicit function Theorem: Express y as a function of x (or x as function of y)!

- Write system as $\max_x f(x, g(x))$

- f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) * \frac{\partial g(x)}{\partial x} = 0$

- What is $\frac{\partial g(x)}{\partial x}$?

- Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) * (-h'_x/h'_y) = 0$ or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$

- **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

$$\max_{x_1,\ldots,x_n} f\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right)$$

$$\text{s.t.} \quad \begin{cases} h_1\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \\ h_2\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \\ \ldots \\ h_m\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \end{cases}$$

with $n > m$. Let $\mathbf{x}^* = \mathbf{x}^*(\mathbf{p})$ be a local solution to this problem.

- Assume:
  - f and h differentiable at x^*
  - the following Jacobian matrix at $\mathbf{x}^*$ has maximal rank

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \ldots & \ldots & \ldots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Then, there exists a vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_m)$ such that $(\mathbf{x}^*, \boldsymbol{\lambda})$ maximize the Lagrangean function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}; \mathbf{p}) - \sum_{j=0}^{m} \lambda_j h_j(\mathbf{x}; \mathbf{p})$$

- Case $n = 2, m = 1$.

- First order conditions are

$$\frac{\partial f(\mathbf{x}; \mathbf{p})}{\partial x_i} - \lambda \frac{\partial h(\mathbf{x}; \mathbf{p})}{\partial x_i} = 0$$

for $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$

- **Constrained Maximization, Sufficient condition for the case $n = 2, m = 1$.**

- If $\mathbf{x}^*$ satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix} 0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial^2 x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_2}(\mathbf{x}^*) \end{pmatrix}$$

  is positive, then $\mathbf{x}^*$ is a constrained maximum.

- If it is negative, then $\mathbf{x}^*$ is a constrained minimum.

- Why? This is just the Hessian of the Lagrangean L with respect to λ, x_1, and x_2

- Example 4: $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

- F.o.c. with respect to x:

- F.o.c. with respect to y:

- F.o.c. with respect to λ:

- Candidates to solution?

- Maxima and minima?

3 Envelope Theorem II

- Envelope Theorem II: Ch. 2, pp. 42-43 (44, 9th Ed)

- **Envelope Theorem for Constrained Maximization.** In problem above consider $F(p) \equiv f(\mathbf{x}^*(\mathbf{p});\mathbf{p})$. We are interested in $dF(p)/dp$. We can neglect indirect effects:

$$\frac{dF}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p});\mathbf{p})}{\partial p_i} - \sum_{j=0}^{m} \lambda_j \frac{\partial h_j(\mathbf{x}^*(\mathbf{p});\mathbf{p})}{\partial p_i}$$

- Example 4 (continued). $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $df(x^*(p), y^*(p))/dp$?

- Envelope Theorem.

4 Preferences

- Part 1 of our journey in microeconomics: *Consumer Theory*

- Choice of consumption bundle:

  1. Consumption today or tomorrow
  2. work, study, and leisure
  3. choice of government policy

- Starting point: preferences.

  1. 1 egg today $\succ$ 1 chicken tomorrow
  2. 1 hour doing problem set $\succ$ 1 hour in class $\succ$ ... $\succ$ 1 hour out with friends
  3. War on Iraq $\succ$ Sanctions on Iraq

5 Next Class

- Properties of Preferences
- From Preferences to Utility
- Common Utility Functions