Economics 101A
(Lecture 5)

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Outline

1. Envelope Theorem II

2. Properties of Preferences (continued)

3. From Preferences to Utility (and viceversa)

4. Common Utility Functions

5. Utility maximization

1 Envelope Theorem II

- Envelope Theorem II: Ch. 2, pp. 44 [OLD, 46-47]

- **Envelope Theorem for Constrained Maximization.** In problem above consider $F(p) \equiv f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})$. We are interested in $dF(p)/dp$. We can neglect indirect effects:

$$\frac{dF}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i} - \sum_{j=0}^{m} \lambda_j \frac{\partial h_j(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i}$$

- Example 4 (continued). $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $df(x^*(p), y^*(p))/dp$?

- Envelope Theorem.

2 Properties of Preferences

- Nicholson, Ch. 3, pp. 69-70 [OLD: 66]

- Commodity set X (apples vs. strawberries, work vs. leisure, consume today vs. tomorrow)

- Preference relation $\succeq$ over X

- A preference relation $\succeq$ is *rational* if

  1. It is *complete*: For all x and y in X, either $x \succeq y$, or $y \succeq x$ or both

  2. It is *transitive*: For all x, y, and z, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$

- Preference relation $\succeq$ is *continuous* if for all y in X, the sets $\{x : x \succeq y\}$ and $\{x : y \succeq x\}$ are closed sets.

- Example: $X = R^2$ with map of indifference curves

- Counterexamples:

  1. Incomplete preferences. Dominance rule.

  2. Intransitive preferences. Quasi-discernible differences.

  3. Discontinuous preferences. Lexicographic order

- Other features of preferences

- Preference relation $\succeq$ is:

  - *monotonic* if $x \geq y$ implies $x \succeq y$.

  - *strictly monotonic* if $x \geq y$ and $x_j > y_j$ for some j implies $x \succ y$.

  - *convex* if for all x, y, and z in X such that $x \succeq z$ and $y \succeq z$, then $tx + (1 - t)y \succeq z$ for all t in $[0, 1]$

3 From preferences to utility

- Nicholson, Ch. 3

- Economists like to use utility functions $u : X \to R$

- $u(x)$ is 'liking' of good x

- $u(a) > u(b)$ means: I prefer a to b.

- **Def.** Utility function u represents preferences $\succeq$ if, for all x and y in X, $x \succeq y$ if and only if $u(x) \geq u(y)$.

- **Theorem.** If preference relation $\succeq$ is rational and continuous, there exists a continuous utility function $u : X \to R$ that represents it.

- [Skip proof]

- Example:

$$(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2$$

- Draw:

- Utility function that represents it: $u(x) = x_1 + x_2$

- But... Utility function representing $\succeq$ is not unique

- Take $3u(x)$ or $\exp(u(x))$

- $u(a) > u(b) \Longleftrightarrow \exp(u(a)) > \exp(u(b))$

- If $u(x)$ represents preferences $\succeq$ and f is a strictly increasing function, then $f(u(x))$ represents $\succeq$ as well.

- If preferences are represented from a utility function, are they rational?
  - completeness
  - transitivity

- Indifference curves: $u(x_1, x_2) = \bar{u}$

- They are just implicit functions! $u(x_1, x_2) - \bar{u} = 0$

$$\frac{dx_2}{dx_1} = -\frac{U'_{x_1}}{U'_{x_2}} = MRS$$

- Indifference curves for:
  - monotonic preferences;
  - strictly monotonic preferences;
  - convex preferences

4 Common utility functions

- Nicholson, Ch. 3, pp. 82-86 [OLD: 80–84]

1. Cobb-Douglas preferences: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$
    - $MRS = -\alpha x_1^{a-1} x_2^{1-\alpha} / (1-a) x_1^{\alpha} x_2^{-\alpha} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$

2. Perfect substitutes: $u\left(x_1, x_2\right) = \alpha x_1 + \beta x_2$
    - $MRS = -\alpha / \beta$

3. Perfect complements: $u(x_1, x_2) = \min(\alpha x_1, \beta x_2)$

   - MRS discontinuous at $x_2 = \frac{\alpha}{\beta} x_1$

4. Constant Elasticity of Substitution: $u(x_1, x_2) = \left(\alpha x_1^\rho + \beta x_2^\rho\right)^{1/\rho}$

   - $MRS = -\frac{\alpha}{\beta}\left(\frac{x_1}{x_2}\right)^{\rho-1}$
   - if $\rho = 1$, then...
   - if $\rho = 0$, then...
   - if $\rho \to -\infty$, then...

5 Utility Maximization

- Nicholson, Ch. 4, pp. 94–105 [OLD: 91–103]

- $X = R_+^2$ (2 goods)

- Consumers: choose bundle $x = (x_1, x_2)$ in X which yields highest utility.

- Constraint: income $= M$

- Price of good 1 $= p_1$, price of good 2 $= p_2$

- Bundle x is feasible if $p_1 x_1 + p_2 x_2 \leq M$

- Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t.\ p_1 x_1 + p_2 x_2 \leq M$$
$$x_1 \geq 0,\ x_2 \geq 0$$

- Maximization subject to inequality. How do we solve that?

- Trick: u strictly increasing in at least one dimension. ($\succeq$ strictly monotonic)

- Budget constraint always satisfied with equality

- Ignore temporarily $x_1 \geq 0$, $x_2 \geq 0$ and check afterwards that they are satisfied for x_1^* and x_2^*.

- Problem becomes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t.\ p_1 x_1 + p_2 x_2 - M = 0$$

- $L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 = M)$

- F.o.c.s:

$$\begin{aligned} u'_{x_i} - \lambda p_i &= 0 \text{ for } i = 1, 2 \\ p_1 x_1 + p_2 x_2 - M &= 0 \end{aligned}$$

- Moving the two terms across and dividing, we get:

$$MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}$$

- Graphical interpretation.

- Example with CES utility function.

$$\max_{x_1,x_2} \left(\alpha x_1^\rho + \beta x_2^\rho\right)^{1/\rho}$$
$$s.t.\ p_1x_1 + p_2x_2 - M = 0$$

- Lagrangean =

- F.o.c.:

- Special case: $\rho = 0$ (Cobb-Douglas)

6 Next Class

- Utility Maximization – tricky cases

- Indirect Utility Function

- Comparative Statics:

  - with respect to price

  - with respect to income