

Economics 101A

(Lecture 4)

Stefano DellaVigna

September 7, 2006

Outline

1. Constrained Maximization
2. Envelope Theorem II
3. Preferences
4. Properties of Preferences
5. From Preferences to Utility

1 Constrained Maximization

- Idea: Use implicit function theorem.
- Heuristic solution of system

$$\begin{aligned} \max_{x,y} f(x, y) \\ \text{s.t. } h(x, y) = 0 \end{aligned}$$

- Assume:
 - continuity and differentiability of h
 - $h'_y \neq 0$ (or $h'_x \neq 0$)
- Implicit function Theorem: Express y as a function of x (or x as function of y)!

- Write system as $\max_x f(x, g(x))$
- f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) * \frac{\partial g(x)}{\partial x} = 0$
- What is $\frac{\partial g(x)}{\partial x}$?
- Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) * (-h'_x/h'_y) = 0$ or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$

- **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

$$\begin{array}{l} \max_{x_1, \dots, x_n} f(x_1, x_2, \dots, x_n; \mathbf{p}) \\ \text{s.t.} \quad \left\{ \begin{array}{l} h_1(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ h_2(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ \dots \\ h_m(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \end{array} \right. \end{array}$$

with $n > m$. Let $\mathbf{x}^* = \mathbf{x}^*(\mathbf{p})$ be a local solution to this problem.

- Assume:
 - f and h differentiable at x^*
 - the following Jacobian matrix at \mathbf{x}^* has maximal rank

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \dots & \dots & \dots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Then, there exists a vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ such that $(\mathbf{x}^*, \boldsymbol{\lambda})$ maximize the Lagrangean function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}; \mathbf{p}) - \sum_{j=0}^m \lambda_j h_j(\mathbf{x}; \mathbf{p})$$

- Case $n = 2, m = 1$.
- First order conditions are

$$\frac{\partial f(\mathbf{x}; \mathbf{p})}{\partial x_i} - \lambda \frac{\partial h(\mathbf{x}; \mathbf{p})}{\partial x_i} = 0$$

for $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$

- **Constrained Maximization, Sufficient condition for the case $n = 2, m = 1$.**

- If \mathbf{x}^* satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix} 0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1^2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2^2}(\mathbf{x}^*) \end{pmatrix}$$

is positive, then \mathbf{x}^* is a constrained maximum.

- If it is negative, then \mathbf{x}^* is a constrained minimum.
- Why? This is just the Hessian of the Lagrangean L with respect to λ , x_1 , and x_2

- Example 4: $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

- F.o.c. with respect to x :

- F.o.c. with respect to y :

- F.o.c. with respect to λ :

- Candidates to solution?

- Maxima and minima?

2 Envelope Theorem II

- Envelope Theorem II: Ch. 2, pp. 44 [OLD, 46-47]
- **Envelope Theorem for Constrained Maximization.** In problem above consider $F(p) \equiv f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})$. We are interested in $dF(p)/dp$. We can neglect indirect effects:

$$\frac{dF}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i} - \sum_{j=0}^m \lambda_j \frac{\partial h_j(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i}$$

- Example 4 (continued). $\max_{x,y} x^2 - xy + y^2$ s.t.
 $x^2 + y^2 - p = 0$
- $df(x^*(p), y^*(p))/dp?$
- Envelope Theorem.

3 Preferences

- Part 1 of our journey in microeconomics: *Consumer Theory*
- Choice of consumption bundle:
 1. Consumption today or tomorrow
 2. work, study, and leisure
 3. choice of government policy
- Starting point: preferences.
 1. 1 egg today \succ 1 chicken tomorrow
 2. 1 hour doing problem set \succ 1 hour in class \succ ... \succ 1 hour out with friends
 3. War on Iraq \succ Sanctions on Iraq

4 Properties of Preferences

- Nicholson, Ch. 3, pp. 69-70 [OLD: 66]
- Commodity set X (apples vs. strawberries, work vs. leisure, consume today vs. tomorrow)
- Preference relation \succeq over X
- A preference relation \succeq is *rational* if
 1. It is *complete*: For all x and y in X , either $x \succeq y$, or $y \succeq x$ or both
 2. It is *transitive*: For all x , y , and z , $x \succeq y$ and $y \succeq z$ implies $x \succeq z$
- Preference relation \succeq is *continuous* if for all y in X , the sets $\{x : x \succeq y\}$ and $\{x : y \succeq x\}$ are closed sets.

- Example: $X = \mathbb{R}^2$ with map of indifference curves

- Counterexamples:

1. Incomplete preferences. Dominance rule.

2. Intransitive preferences. Quasi-discernible differences.

3. Discontinuous preferences. Lexicographic order

- Indifference relation \sim : $x \sim y$ if $x \succeq y$ and $y \succeq x$
- Strict preference: $x \succ y$ if $x \succeq y$ and not $y \succeq x$
- Exercise. If \succeq is rational,
 - \succ is transitive
 - \sim is transitive
 - Reflexive property of \succeq . For all x , $x \succeq x$.

- Other features of preferences

- Preference relation \succeq is:
 - *monotonic* if $x \succeq y$ implies $x \succ y$.

 - *strictly monotonic* if $x \succeq y$ and $x_j > y_j$ for some j implies $x \succ y$.

 - *convex* if for all x, y , and z in X such that $x \succeq z$ and $y \succeq z$, then $tx + (1 - t)y \succeq z$ for all t in $[0, 1]$

5 From preferences to utility

- Nicholson, Ch. 3
- Economists like to use utility functions $u : X \rightarrow R$
- $u(x)$ is 'liking' of good x
- $u(a) > u(b)$ means: I prefer a to b .
- **Def.** Utility function u represents preferences \succeq if, for all x and y in X , $x \succeq y$ if and only if $u(x) \geq u(y)$.
- **Theorem.** If preference relation \succeq is rational and continuous, there exists a continuous utility function $u : X \rightarrow R$ that represents it.

- [Skip proof]

- Example:

$$(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2$$

- Draw:

- Utility function that represents it: $u(x) = x_1 + x_2$

- But... Utility function representing \succeq is not unique

- Take $3u(x)$ or $\exp(u(x))$

- $u(a) > u(b) \iff \exp(u(a)) > \exp(u(b))$

- If $u(x)$ represents preferences \succeq and f is a strictly increasing function, then $f(u(x))$ represents \succeq as well.

- If preferences are represented from a utility function, are they rational?
 - completeness
 - transitivity

- Indifference curves: $u(x_1, x_2) = \bar{u}$
- They are just implicit functions! $u(x_1, x_2) - \bar{u} = 0$

$$\frac{dx_2}{dx_1} = -\frac{U'_{x_1}}{U'_{x_2}} = MRS$$

- Indifference curves for:
 - monotonic preferences;
 - strictly monotonic preferences;
 - convex preferences

6 Next Class

- Common Utility Functions
- Utility Maximization