

Economics 101A (Lecture 2)

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Outline

1. Multivariate optimization
2. Comparative Statics
3. Implicit function theorem
4. Envelope Theorem

1 Multivariate optimization

- Nicholson, Ch.2, pp. 26–32
- Function from R^n to R : $y = f(x_1, x_2, \dots, x_n)$
- Partial derivative with respect to x_i :

$$\frac{\partial f(x_1, \dots, x_n)}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

- Slope along dimension i
- Total differential:

$$df = \frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2 + \dots + \frac{\partial f(x)}{\partial x_n} dx_n$$

- One important economic example
- Example 1: Partial derivatives of $y = f(L, K) = L^{.5}K^{.5}$
- $f'_L =$
(marginal productivity of labor)
- $f'_K =$
(marginal productivity of capital)
- $f''_{L,K} =$

Maximization over an open set (like R)

- **Necessary condition for maximum** x^* is

$$\frac{\partial f(x^*)}{\partial x_i} = 0 \quad \forall i \quad (1)$$

or in vectorial form

$$\nabla f(x) = 0$$

- These are commonly referred to as first order conditions (f.o.c.)

- Sufficient conditions? Define Hessian matrix H :

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} & \cdots & f''_{x_1,x_n} \\ \cdots & \cdots & \cdots & \cdots \\ f''_{x_n,x_1} & f''_{x_n,x_2} & \cdots & f''_{x_n,x_n} \end{pmatrix}$$

- Subdeterminant $|H|_i$ of Matrix H is defined as the determinant of submatrix formed by first i rows and first i columns of matrix H .
- Examples.
 - $|H|_1$ is determinant of f''_{x_1,x_1} , that is, f''_{x_1,x_1}
 - $|H|_2$ is determinant of

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} \\ f''_{x_2,x_1} & f''_{x_2,x_2} \end{pmatrix}$$

- **Sufficient condition for maximum x^* .**
 1. x^* must satisfy first order conditions;
 2. Subdeterminants of matrix H must have alternating signs, with subdeterminant of H_1 negative.

- Case with $n = 2$
- Condition 2 reduces to $f''_{x_1, x_1} < 0$ and $f''_{x_1, x_1} f''_{x_2, x_2} - (f''_{x_1, x_2})^2 > 0$.
- Example 2: $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$
- First order condition w/ respect to x_1 ?
- First order condition w/ respect to x_2 ?
- $x_1^*, x_2^* =$
- For which p_1, p_2 is it a maximum?
- For which p_1, p_2 is it a minimum?

2 Comparative statics

- Economics is all about 'comparative statics'
- What happens to optimal economic choices if we change one parameter?
- Example: Car production. Consumer:
 1. Car purchase and increase in oil price
 2. Car purchase and increase in income
- Producer:
 1. Car production and minimum wage increase
 2. Car production and decrease in tariff on Japanese cars
- Next two sections

3 Implicit function theorem

- Implicit function: Ch. 2, pp. 32–33 [OLD, 32–34]
- Consider function $x_2 = g(x_1, p)$
- Can rewrite as $x_2 - g(x_1, p) = 0$
- **Implicit function** has form: $h(x_2, x_1, p) = 0$
- Often we need to go from implicit to explicit function

- Example 3: $1 - x_1 * x_2 - e^{x_2} = 0$.
- Write x_1 as function of x_2 :
- Write x_2 as function of x_1 :

- **Univariate implicit function theorem (Dini):** Consider an equation $f(p, x) = 0$, and a point (p_0, x_0) solution of the equation. Assume:
 1. f continuous and differentiable in a neighbourhood of (p_0, x_0) ;
 2. $f'_x(p_0, x_0) \neq 0$.
- Then:
 1. There is one and only function $x = g(p)$ defined in a neighbourhood of p_0 that satisfies $f(p, g(p)) = 0$ and $g(p_0) = x_0$;
 2. The derivative of $g(p)$ is

$$g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}$$

- Example 3 (continued): $1 - x_1 * x_2 - e^{x_2} = 0$
- Find derivative of $x_2 = g(x_1)$ implicitly defined for $(x_1, x_2) = (1, 0)$
- Assumptions:
 1. Satisfied?
 2. Satisfied?
- Compute derivative

● **Multivariate implicit function theorem (Dini):**

Consider a set of equations $(f_1(p_1, \dots, p_n; x_1, \dots, x_s) = 0; \dots; f_s(p_1, \dots, p_n; x_1, \dots, x_s) = 0)$, and a point (p_0, x_0) solution of the equation. Assume:

1. f_1, \dots, f_s continuous and differentiable in a neighbourhood of (p_0, x_0) ;
2. The following Jakobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ evaluated at (p_0, x_0) has determinant different from 0:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_s} \\ \dots & \dots & \dots \\ \frac{\partial f_s}{\partial x_1} & \dots & \frac{\partial f_s}{\partial x_s} \end{pmatrix}$$

• Then:

1. There is one and only set of functions $x = \mathbf{g}(p)$ defined in a neighbourhood of p_0 that satisfy $\mathbf{f}(p, \mathbf{g}(p)) = \mathbf{0}$ and $\mathbf{g}(p_0) = x_0$;
2. The partial derivative of x_i with respect to p_k is

$$\frac{\partial g_i}{\partial p_k} = - \frac{\det \left(\frac{\partial(f_1, \dots, f_s)}{\partial(x_1, \dots, x_{i-1}, p_k, x_{i+1}, \dots, x_s)} \right)}{\det \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)}$$

- Example 2 (continued): Max $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$
- f.o.c. $x_1 : 2p_1 * x_1 - 2 = 0 = f_1(p, x)$
- f.o.c. $x_2 : 2p_2 * x_2 - 5 = 0 = f_2(p, x)$
- Comparative statics of x_1^* with respect to p_1 ?
- First compute $\det \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- Then compute $\det \left(\frac{\partial(f_1, \dots, f_s)}{\partial(x_1, \dots, x_{i-1}, p_k, x_{i+1}, \dots, x_s)} \right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- Finally, $\frac{\partial x_1}{\partial p_1} =$

- Why did you compute $\det \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)$ already?

4 Envelope Theorem

- Ch. 2, pp. 33–37 [OLD, 34–39]
- You now know how x_1^* varies if p_1 varies.
- How does $h(\mathbf{x}^*(\mathbf{p}))$ vary as p_1 varies?
- Differentiate $h(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)$ with respect to p_1 :

$$\begin{aligned} & \frac{dh(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)}{dp_1} \\ = & \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_1} * \frac{\partial x_1^*(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \\ & + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_2} * \frac{\partial x_2^*(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \\ & + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \end{aligned}$$

- Notice: First two terms are zero.

- **Envelope Theorem** for unconstrained maximization. Assume that you maximize function $f(\mathbf{x}; \mathbf{p})$ with respect to x . Consider then the function f at the optimum, that is, $f(\mathbf{x}^*(\mathbf{p}), \mathbf{p})$. The total differential of this function with respect to p_i equals the partial derivative with respect to p_i :

$$\frac{df(\mathbf{x}^*(\mathbf{p}), \mathbf{p})}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}), \mathbf{p})}{\partial p_i}.$$

- You can disregard the indirect effects. Graphical intuition.

5 Next Class

- Next class:
 - Convexity and Concavity
 - Constrained Maximization
 - Envelope Theorem II

- Going toward:
 - Preferences
 - Utility Maximization (where we get to apply maximization techniques the first time)