Preliminary and Incomplete Draft

Testing the Semiparametric Box-Cox Model with the Bootstrap

N.E. Savin, Henry B. Tippie College of Business, Department of Economics, University of Iowa, 108 John Pappajohn Business Bldg., Iowa City, IA 52242-1000. Tel: (319) 335-0855, FAX (319) 335-1956, gene-savin@uiowa.edu.

Allan H. Würtz, Department of Economics, University of Aarhus, Building 350, DK 8000 Aarhus C, Denmark. Tel: 45-8942-1507; FAX:45-8613-6334 awurtz@econ.dk.

July 23, 2001

Abstract

This paper considers testing the transformation parameter of the Box-Cox model when the distribution of the error is unknown. The transformation parameter indexes the most commonly used functional forms. The null hypothesis is tested using Wald and Lagrange Multiplier (LM) statistics constructed from GMM estimators. The finite sample performance of the tests with asymptotic and bootstrap critical values is investigated in a Monte Carlo study. The LM test with asymptotic critical values satisfactorily controls the Type I error for sample sizes available in practice. The numerical performance of the Wald test with bootstrap critical values is disappointing.

JEL Classification: C13, C14

Field Designations: 17

1.Introduction

The Box-Cox (1964) regression model is a transformation model of the form

$$T(Y, a) = X'b + U, \tag{1.1}$$

where T is a strictly increasing function, Y is an observed dependent variable, X is an observed K dimensional random column vector, b is a vector of constant parameters that is conformable with X, and U is an unobserved random variable.

The Box-Cox transformation is

$$T(y, a) = \begin{cases} \dfrac{y^{a} - 1}{a}, & if\, a \neq 0, y \geq 0, \\ \log y, & if\, a = 0, y \geq 0. \end{cases} \tag{1.2}$$

The transformation provides a flexible parameterization of the relation between the dependent variable and the regressors. In particular, the model is a linear model if $\alpha = 1$, a power transformation model if $\alpha \neq 0$ or 1, and a log linear model if $\alpha = 0$.

If the cumulative distribution function (CDF) of U, denoted by F, is known or known up to finite dimensional parameters, then a and b and any parameters of F can be estimated by maximum likelihood. A widely used procedure, which was suggested by Box and Cox (1964), is to estimate a and b by the maximum likelihood (ML) under the assumption that the U is normally distributed. The resulting estimator of a and b is referred to as the Box-Cox ML estimator. The Box-Cox ML estimator is discussed in many econometric textbooks, for example, Amemiya (1985), Greene (2000) and Mittelhammer, Judge and Miller (2000) and Ruud (2000).

The assumption of normality cannot be strictly true, however. $T(y,a)$ is bounded from below (above) if $a > 0$ ($a < 0$) unless a is an odd integer or 0. Thus, the Box-Cox transformation cannot be applied to models in which the dependent variable can be negative or the distribution of U has unbounded support, and, hence, this rules out the case where U is normally distributed.

In practice, however, F is often unknown. Thus, an empirically relevant statistical problem is to obtain consistent estimators of a and b when F is unknown. The solution proposed by Amemiya and Powell (1981) is to use the nonlinear two-stage least squares (NL2SLS) estimator of a and b. The NL2SLS estimator is a generalized method of moments (GMM) estimator, and it is the efficient GMM estimator for the choice of instruments used by Amemiya and Powell, provided U is homoskedastic. Horowitz (1998) discusses GMM estimation of a and b .

Khazzoom (1989) pointed out that the NL2SLS estimates for this model are ill-defined for data sets in which the dependent variable always exceeds (or is exceeded by) one. The non-negative GMM objective function has a global minimum of zero as a tends to minus infinity when y >1 and infinity when y <1. Powell (1996) has proposed a simple rescaling of the GMM objective function that helps ensure the estimates are interior points of the parameter space.

The focus of this paper is on testing the transformation parameter a in the Box-Cox model when F is unknown. This null is tested using Wald and Lagrange Multiplier (LM) test statistics constructed from GMM estimators. The test of the null is based on an estimator of the *Type I critical value*. Horowitz and Savin (2000) define this critical value as one that would be obtained if the exact finite sample distribution of the test statistic under the true data generation process were known. In our setting, the true Type I critical value is unknown because the null hypothesis is composite; that is, the exact finite-sample distribution of the test statistic depends on b and F, population parameters not specified by the null. Thus, an approximation to the Type I critical value is required to implement the test.

An approximation to the Type I error critical value can be obtained by using the first-order asymptotic distribution of the test statistic to approximate its finite-sample distribution. The approximation is useful because most test statistics in econometrics are asymptotically pivotal: their asymptotic distributions do not depend on unknown population parameters when the null hypothesis being tested is true. Hence, an approximate Type I critical value can be obtained from first-order asymptotic distribution theory without knowledge of the true data generation process. This is true for the Wald and LM statistics employed to test null hypotheses about the transformation parameter.

However, Monte Carlo experiments have shown that first-order asymptotic theory often gives a poor approximation to the exact distributions of test statistics with the sample sizes available in applications. As a result, the difference between the true and nominal probabilities that a test rejects a correct hypothesis, the error in the rejection probability, can be large when an asymptotic critical value is used.

Another approach is to use the bootstrap. The bootstrap is a method for estimating the distribution of a statistic or a feature of the distribution, such as a moment or a quantile. Under regularity conditions, the bootstrap provides an approximation to the Type I critical value that is more accurate than the approximation of first-order asymptotic theory. These regularity conditions are satisfied for the Wald and LM test statistics considered in this paper.

This paper investigates the numerical performance of the tests with asymptotic critical values and with bootstrap critical values. In the Monte Carlo experiments, two different specifications are considered for the Box-Cox model. The first is the two-parameter gamma distribution for *Y* given x proposed by Amemiya and Powell (1981), and the second is the truncated normal for $T(y,a)$ given x used by Poirier (1978). The Monte Carlo results show that the tests with bootstrap critical values provide good control over the Type I error for sample sizes used in applications.

In the context of the Box-Cox model, the linear model can be tested against other specifications that are indexed by the transformation parameter. For example, the linear model can be tested against the log-linear model by testing the null hypothesis that $a = 1$ against the alternative $a = 0$. For the tests to be useful, they must be able to discriminate between alternative specifications. We examine the power of the bootstrap-based Wald test against various alternative specifications and similarly for the LM test.

The organization of the paper is the following. GMM estimation of the Box-Cox model is reviewed in Section 2. Section 3 introduces the Wald and LM tests constructed from GMM estimators. The bootstrap method is presented in Section 4. Section 5 describes the main features of the design of the experiments. The results of the Monte Carlo experiments on the numerical performance of the bootstrap are presented in Section 6. Section 7 contains the concluding comments.

2. GMM Estimators

This section reviews the estimation of the parameters a and b by the general method of moments (GMM). The GMM estimator is consistent under weak distribution assumptions (Mittlehammer et al. (2000)).

Let W be a column vector of valid instruments. Validity requires that $E(W'U) = 0$ and dim(W) = m $\geq$ k + 1. Powers, crossproducts and other nonlinear functions of X can be used to form W. Given W, the parameters a and b can be estimated using the population moment condition

$$E\{W[T(Y,a) - X'b]\} = 0, \tag{2.1}$$

provided that this equation uniquely determines a and b.

Denote the estimation data by $\{Y_i, X_i$: i = 1, …, n,$\}$ and assume that they are a random sample from the joint distribution of $\{Y, X\}$. . Let $q = (a, b')'$,

$U_i(q) = T(Y_i, a) - X_i' b$ and $U(q) = (U_1(q), \ldots, U_n(q))'$. Also let W = $[W_1, \ldots, W_n]'$ denote the matrix of instruments where W_i is a vector of functions of X_i that includes X_i. Finally, let $\hat{q}_n = (\hat{a}_n, \hat{b}_n')'$ where $\hat{a}_n$ and $\hat{b}_n$ denote the GMM estimators of a and b, respectively.

The GMM estimator solves

$$\underset{q}{minimize}: S_n(q) = U(q)'\mathrm{W}\Omega\mathrm{W}'U(q) \tag{2.2}$$

where Ω_n is a positive definite, possibly stochastic matrix. One possible choice of Ω_n is $\Omega_n = [\mathrm{W}'\mathrm{W}]^{-1}$, in which case (2.2) gives the NL2SLS estimator of Amemiya (1974, 1985). This choice is asymptotically efficient if the errors U_i are homoskedastic. Amemiya and Powell (1981) and Amemiya (1985) discuss the use of NL2SLS for estimation of the Box-Cox model.

We note that change in the NL2SLS estimate of b due to a rescaling of X is the same as the change in the ordinary least squares (OLS) estimate in the linear regression model. By contrast, the effect of rescaling Y depends on whether the parameters are exactly or overidentified. In the exactly identified case, rescaling Y has no effect on the NL2SLS estimate of a; only b is affected. In the overidentified case, rescaling Y changes the estimates of both a and b.

The consistency of the estimator minimizing (2.3) is established by verification of three conditions: compactness of the parameter space; convergence in probability of the objective function S_n to its expected value, uniformly in a and b; and uniqueness of the solutions satisfying the moment condition (2.1). The compactness and identification conditions turn out to be demanding due to the nature of the transformation function, $T(Y, a)$.

As Khazzoom (1989) notes, if y > 1, then $T(y,a) \to 0$ as $a \to -\infty$, and, similarly, if y < 1, $T(y,a) \to 0$ as $a \to \infty$. This implies that the compactness plays a crucial role in the uniqueness of the solution of (2.1). In particular, each residual $U_i(q) = T(y_i, a) - x_i^{'} b$ can be set equal to 0 by setting $a = -\infty$ and $b = 0$ if each y_i >1. The resulting pathology of the objective function is important in practice since in many data sets all values of the dependent variable exceed one.

To avoid the problem associated with the scaling of the dependent variable, Powell (1996) suggested the following rescaling of the GMM objective function:

$$Q_n(q) = S_n(q) \cdot (\dot{y})^{-2a}, \tag{2.3}$$

where the GMM objective function S_n is given in (2.2) and $\dot{y}$ is the geometric mean of the absolute values of the dependent variable:

$$\dot{y} \equiv \exp\left\{\frac{1}{n}\sum_{i=1}^{n}\log(|\, y_i\,|)\right\}. \tag{2.4}$$

The rescaled GMM objective function Q_n is less likely than S_n to be minimized by values on the boundary of the parameter space. However, as Powell (1996) notes, rescaling the original GMM function by $\dot{y}^{-2a}$ cannot guarantee that a unique and finite minimizing value of a will exist.

The objective function

$$Q_n(q) = [\,U(q)'\mathrm{W} / \dot{y}^{a}\,]'\,\Omega\,[\mathrm{W}'U(q) / \dot{y}^{a}\,] \tag{2.5}$$

can be concentrated as a function of a only. This implies that for a given a the optimal b in (2.7) is

$$b(a) = [(\sum_{i=1}^{n} W_i X_i')'\Omega(\sum_{i=1}^{n} W_i X_i')]^{-1}(\sum_{i=1}^{n} W_i X_i')'\Omega\sum_{i=1}^{n} W_i (T(y_i, a) \tag{2.6}$$

since $\dot{y}^{-a}$ cancels. The concentrated objective function in a is obtained by substituting (2.8) into (2.7), which gives

$$Q_n(a) = Q_n(a, b(a)) = S_n(a, b(a)) / \dot{y}^{2^{a}} = S_n(a) / \dot{y}^{2a}\,, \tag{2.7}$$

where $S_n(a)$ is the concentrated objective function for GMM. The estimation procedure for rescaled GMM simplifies to a one-dimensional grid search via the concentrated objective function and similarly for the original GMM estimation problem. Note that if NL2SLS and RNL2SLS give the same estimate of a, then they both give the same estimate of b.

Powell (1996) argues that the original and rescaled GMM estimators have the same asymptotic distribution. Hence, the standard formulae for the first-order asymptotic distribution and asymptotic covariance matrix estimators for GMM estimators apply directly to the rescaled estimators.

3. Tests

This section introduces the Wald and LM tests of the null hypotheses h(q) = 0 where h(q) is a q dimensional differentiable function. The tests are constructed using GMM estimators.

Hansen (1982) derived the asymptotic distributional properties of the GMM estimator. Hansen (1982) showed under mild regularity conditions that $q_n = (\hat{a}_n, \hat{b}_n')'$ is a consistent estimator of q and that q_n is asymptotically normally distributed:

$$n^{1/2}(\hat{q}_n - q) \to^d N(0,V) \tag{3.1}$$

where

$$V = (D'\Omega D)^{-1}, \tag{3.2}$$

with $D = E\frac{\partial}{\partial q}W[T(Y,a) - X b]$ and $\Omega = p\lim_{n\to\infty}\Omega_n$. Letting $U_q = \partial U(q)/\partial q$ and,

$\hat{U}_q = \partial U(\hat{q}_n)/\partial q$, V can be estimated by replacing D in (3.2) by $W'\hat{U}_q$ and Ω by Ω_n.

Thus, (3.1) and (3.2) with V replaced by

$$\hat{V}_n = \hat{U}_q' W\Omega_n W'\hat{U}_q$$

make it possible to carry out inference in sufficiently large samples.

The Wald statistic for testing $h(q) = 0$ is based on the unconstrained GMM estimator of q. The Wald statistic is

$$Wald = n \cdot h(\hat{q}_n)'[\hat{h}_q \hat{V}_n \hat{h}_q{}']^{-1} h(\hat{q}_n), \tag{3.3}$$

where $\hat{h}_q = \partial h(\hat{q}_n)/\partial q$. This statistic is distributed asymptotically as a chi-square variable with q degrees of freedom if the null is true. The principle disadvantage of the GMM based Wald statistic is that it is not invariant to reparametrization of the null hypothesis or rescaling of the dependent variable. Spitzer (1984) has shown a similar lack of invariance for the Wald statistic based on the Box-Cox ML estimator; see also Drucker (2000).

The null hypothesis tested in this paper specifies the value of the transformation parameter: H_0: $a = a_0$. The Wald statistic for testing H_0: $a = a_0$ is

$$Wald = \frac{n(\hat{a}_n - a_0)^2}{\hat{s}_n^2}, \tag{3.4}$$

where $\hat{s}_n^2$ is the first diagonal element in $\hat{V}_n$. The Wald statistic (3.4) is distributed asymptotically as chi-square variables with one degree of freedom when the null hypothesis is true. The GMM estimators that can be used in computing (3.4) include as special cases the NL2SLS and RNL2SLS estimators.

Newey and West (1987) have developed an LM test based on the constrained GMM estimator. This LM test is presented in Greene (2000). The constrained estimator, denoted by $\tilde{q}_n = (\tilde{a}_n, \tilde{b}_n{}')'$, solves (2.2) subject to the constraint $h(q) = 0$. The GMM-based LM statistic is

$$n \cdot \tilde{U}' P_W \tilde{U}_q [\tilde{U}_q{}' P_W \tilde{U}_q]^{-1} \tilde{U}_q{}' P_W \tilde{U} / \tilde{U}'\tilde{U}, \tag{3.5}$$

where $P_W = W(W'W)^{-1}W'$, $\tilde{U} = U(\tilde{q}_n)$ and $\tilde{U}_q = \partial U(\tilde{q}_n)/\partial q$. This is $n \cdot R^2$ from a regression of $\tilde{U}$ on $P_W\tilde{U}_q$. That is, the LM statistic can be obtained from regressing $\tilde{U}_q$ on W, calculating the predicted value, and then calculating $n \cdot R^2$ from a regression of the restricted residual on these predicted values. The constrained NL2SLS and RNL2SLS estimates of a are the same, and, hence, the constrained NL2SLS and RNL2SLS estimates of b are the same. As a result, the values of the LM statistic for NL2SLS and RNL2SLS are also the same.

The LM test is especially convenient when testing the null hypothesis H_0: $a = a_0$. Note first that by having X_i included in W_i, the constrained estimator is $\tilde{q}_n = (a_0, \tilde{b}_n')'$ where $\tilde{b}$ is the OLS estimator obtained by regressing $T(Y_i, a_0)$ on X_i. Therefore the constrained residual vector $\tilde{U}$ is just the residual vector from the OLS regression of $T(Y_i, a_0)$ on X_i. Also $\tilde{U}_q = \partial U(\tilde{q}_n)/\partial q = [\tilde{T}_a, -X]$, where $\tilde{T}_a = (\partial T(y_1, a_0)/\partial a, ..., \partial T(y_n, a_0)/\partial a)'$ and X = $[X_1, ..., X_n]'$. Furthermore, if X_i included in W_i, then $P_W\tilde{U}_q = [P_X\tilde{T}_a, -X]$. Thus, the LM statistic for testing α can be obtained in three steps as follows:

1. Obtain the OLS residuals from regressing $T(Y_i, a_0)$ on X_i.

2. Obtain the predicted values regressing $T(Y_i, a_0)$ on W_i.

3. Calculate the test statistic as $n \cdot R^2$ from regressing the residuals from 1 on the predicted value for 2 and X_i. This $n \cdot R^2$ is the LM statistic.

The GMM based LM statistic is invariant to reparametrization of the null hypothesis, but not always to the rescaling of the dependent variable. Invariance to rescaling depends on whether the parameters are exactly identified. The LM statistic is invariant to rescaling of the dependent variable in the exactly identified case but not in. the overidentified case.

4. Bootstrap Critical Values

This section explains how the bootstrap is implemented in a simple setting and shows how the bootstrap can be used to obtain a Type I critical value for hypothesis tests. The presentation is based on Horowitz (1999).

In this section, let the data be a random sample of size n from a probability distribution whose CDF is F_0. Denote the data by $\{X_i : 1,\ldots,,n\}$. Let F_0 belong to a finite or infinite-dimensional family of distribution functions, and let F denote a general member of this family. Let $T_n = T_n(X_1,\ldots,X_n)$ be a statistic. Let $G_n(t, F_0) \equiv P(T_n \leq t)$ denote the exact, finite-sample distribution CDF of T_n. Let $G_n(\cdot, F)$ denote the exact CDF of T_n when the data are sampled from the distribution whose CDF is F .

Usually, $G_n(t, F)$ is a different function of t for different distributions F. An exception occurs if $G_n(\cdot, F)$ does not depend on F, in which case T_n is said to be *pivotal*. For example, the t statistic for testing a hypothesis about the sample mean of a normal population is independent of the unknown population under the null hypothesis and, therefore, is pivotal. Pivotal statistics are not available in most econometric applications, however, without making strong distributional assumptions. Therefore, $G_n(\cdot, F)$ usually depends on F, and $G_n(\cdot, F_0)$ cannot be calculated if F_0 is unknown.

First-order asymptotic distribution theory is a widely used method for estimating $G_n(\cdot, F_0)$. The asymptotic distributions of many econometric statistics are standard normal or chi-square, possibly after centering and normalization regardless of the distribution from which the data were sampled. Such statistics are called *asymptotically pivotal*, meaning that their asymptotic distributions do not depend on unknown population parameters. Let $G_\infty(\cdot, F_0)$ denote the asymptotic distribution of T_n. Let $G_\infty(\cdot, F)$ denote the asymptotic CDF of T_n when the data are sampled from the distribution whose CDF is F. If T_n is asymptotically pivotal, then $G_\infty(\cdot, F) \equiv G_\infty(\cdot)$ does not depend on F under the null hypothesis. Therefore, if n is sufficiently large, $G_n(\cdot, F_0)$ can be estimated by $G_\infty(\cdot)$ without knowing F_0. This method for estimating $G_n(\cdot, F_0)$ is often easy to implement, but $G_\infty(\cdot)$ can be a very poor approximation to $G_n(\cdot, F_0)$ with the sample sizes available in applications.

The bootstrap provides an alternative approximation to $G_n(\cdot, F_0)$ or features of $G_n(\cdot, F_0)$ such as its quantiles when F_0 is unknown. Whereas first-order asymptotic approximations replace the unknown distribution function G_n with the known distribution function G_∞, the bootstrap replaces the unknown distribution function F with a known estimator. Let F_n denote the estimator of F_0. Two possible choices of F_n are:

(1) The empirical distribution function (EDF) of the data:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x),$$

where I is the indicator function.

(2) A parametric estimator of F_0.

If the distribution of X is not assumed to belong to a known parametric family, the EDF of X is the most obvious candidate for F_n. In the case of the semiparametric Box-Cox model, the EDF is the estimator of F_0.

Regardless of the choice of F_n, the bootstrap estimator of $G_n(\cdot, F_0)$ is $G_n(\cdot, F_n)$. Usually, $G_n(\cdot, F_n)$ cannot be evaluated analytically. It can, however, be estimated with arbitrary accuracy by carrying out a Monte Carlo simulation in which random samples are drawn from F_n. Thus, the bootstrap is usually implemented by Monte Carlo simulation. The essential characteristic of the bootstrap is the use of F_n to approximate F_0 in $G_n(\cdot, F_0)$, not the method that is used to evaluate $G_n(\cdot, F_n)$.

Now let T_n be statistic for testing a hypothesis H_0 about the sampled population. Assume that under H_0, T_n is asymptotically pivotal and satisfies certain technical conditions. Consider a symmetrical, two-tailed test of H_0. For such a test, H_0 is rejected if $|T_n|$ exceeds a suitable critical value and is accepted otherwise. This test rejects H_0 at the a level if $|T_n| > z_{na/2}$, where $z_{na/2}$, the exact, finite-sample $a/2$-level critical value, is the 1-$a/2$ quantile of the distribution of T_n. The critical value solves the equation

$$G_n(z_{n,a/2}, F_0) - G_n(-z_{n,a/2}, F_0) = 1 - a \ . \tag{4.1}$$

Horowitz and Savin (2000) refer to $z_{na/2}$ as the exact Type I critical value of the test of H_0. Unless T_n is exactly pivotal, however, equation (4.1) cannot be solved in an application because F_0 is unknown. Therefore, the exact, finite-sample critical value cannot be obtained in an application if T_n is not pivotal.

First-order asymptotic approximations obtain a feasible version of (4.1) by replacing G_n by G_∞. Thus, the asymptotic critical value, $z_{\infty a/2}$, solves

$$G_\infty(z_{\infty,a/2}, F_0) - G_\infty(-z_{\infty,a/2}, F_0) = 1 - a \ . \tag{4.2}$$

Assuming G_∞ is the standard normal distribution when T_n is asymptotically pivotal, $z_{\infty a/2}$ can be obtained from the table of standard normal quantiles. It can be shown that the asymptotic critical value approximates the exact finite sample critical value with an error whose size is $O(n^{-1})$.

The bootstrap obtains a feasible version of (4.1) by replacing F_0 with F_n. Thus the bootstrap critical value, $z_{n,a/2}*$, solves

$$G_n(z_{n,a/2}*, F_0) - G_n(-z_{n,a/2}*, F_0) = 1 - a \, . \tag{4.3}$$

Equation (4.3) usually cannot be solved analytically, but $z_{n,a/2}*$ can be estimated with any desired accuracy by Monte Carlo simulation.

The accuracy of the bootstrap critical value as an estimator of the exact finite-sample critical value $z_{na/2}$ is given by

$$z_{n,a/2}* = z_{n,a/2} + O(n^{-3/2}) \tag{4.4}$$

almost surely. Thus, the bootstrap critical value for a symmetrical, two-tailed test differs from the exact, finite-sample critical value by $O(n^{-3/2})$ almost surely. The bootstrap critical value is more accurate than the asymptotic critical value, $z_{\infty a/2}$, whose error is $O(n^{-1})$.

The rejection probability of the test based on T_n when H_0 is true is $P(|T_n| > z_{n,a/2}) = a$ when the test is based on the exact but infeasible Type I critical value. With the asymptotic critical value, the rejection probability is

$$P(|T_n| \geq z_{\infty,a/2}) = a + O(n^{-1}) \, . \tag{4.5}$$

Thus, with the asymptotic critical value, the true and nominal rejection probabilities differ by $O(n^{-1})$. The rejection probability with the bootstrap value is

$$P(|T_n| \geq z_{n,a/2}*) = a + O(n^{-2}) \, . \tag{4.6}$$

Note that $z_{na/2}*$ is a random variable, which complicates the derivation of (4.6). The result (4.6) says the nominal of rejection probability of a symmetrical, two-tailed test with a bootstrap critical value differs from the true rejection probability by $O(n^{-2})$ when the test statistic is asymptotically pivotal. In contrast, the difference between the nominal and true rejection probabilities is $O(n^{-1})$ when the asymptotic critical value is used. For details, see Hall (1992).

Finally, consider the power of a test based on a bootstrap critical value. Suppose that bootstrap samples are generated by a model that satisfies a false H_0, and, therefore, is misspecified relative to the true data-generation process. If H_0 is simple, meaning that it completely specifies the data-generation process, then the bootstrap amounts to Monte Carlo estimation of the exact finite-sample critical value for testing H_0 against the true data generation process. In most applications, including the one in this paper, the null H_0 is composite. That is, it does not specify the value of a finite-or infinite-dimensional "nuisance' parameter y. It can be shown, however, that a test of a composite hypothesis using a bootstrap-based critical value is a higher-order approximation to a certain exact test. The power of the test with a bootstrap critical value is a higher-order approximation to the power of the exact test.

5. Design of Experiments

This section presents the general features of the designs used in the Monte Carlo experiments. The section concludes with a description of how the bootstrap critical values are computed for the Wald and LM tests of the null hypothesis about the transformation parameter.

The model simulated in the experiments is

$$T(Y, a) = b_0 + b_1 X + U \tag{5.1}$$

where X is a scalar random variable. Let $\{Y_i, X_i, i = 1, \ldots, n\}$ be a sample from (Y, X). The instruments used in the NL2SLS and RNL2SLS estimators are those employed by

Amemiya and Powell (1981), namely, 1, X and X^2. Hence, with this set of instruments, the parameters are exactly identified and the NL2SLS is the efficient GMM estimator.

Three different specifications for the conditional distribution of Y given X are considered for the Box-Cox model. The first is the two-parameter gamma distribution for Y given X proposed by Amemiya and Powell (1981). The second is a truncated normal suggested by Poirier (1978). Let U be $N(0, (0.5)^2)$ with left truncation point $U = -1$. The third is an exponential for U with parameter $\lambda = 4$. The truncated normal and exponential distributions of U are generated independently of X. However, the construction of the conditional distribution of Y in the case of gamma implies that U and X are dependent. The values of X are obtained by random sampling the following marginal distributions of X: Uniform [-1, 1], lognormal based on N(0, 1) and exponential with $\lambda = 1$.

The sample sizes investigated are n = 50, 100, 200. For each specific design, the number of Monte Carlo replications is 1000 (to be increased to 5000 in final draft). The grid for a is divided into units of 0.10 and runs from -5.0 to 5.0 about the true value of a. The computations were performed using GAUSS for Windows NT/95, Version 3.2.33.

Each experiment consists of testing the null hypothesis, H_0: $a = a_0$, a_0 = 0.0, 0.5 1.0. The Wald and LM statistics for testing H_0 are described in Section 3. The asymptotic critical value, $z_{\infty a/2}$, for the Wald and LM tests is obtained from a table of the quantiles of the chi-square one distribution, or of the standard normal distribution.

The Monte Carlo procedure for computing the bootstrap critical value for the Wald test is the following:

W1. Use the estimation data $\{Y_i, X_i$: i = 1, … , n,$\}$ to compute the unconstrained GMM estimator.

W2. Generate a bootstrap sample by size n by sampling {*Y*, *X*} pairs from the estimation data with replacement. Compute the unconstrained GMM estimators of q and V from the bootstrap sample. Call the results $\hat{q}_n* = (\hat{a}_n*, \hat{b}_n*')'$ and $\hat{V}_n*$. The bootstrap version of the Wald statistic is

$$Wald* = \frac{n(a_n* - \hat{a}_n)^2}{\hat{s}_n^2*} \qquad (5.2)$$

where $\hat{s}_n^2*$ is the first element of $\hat{V}_n*$. Note that a_0 is replaced by $\hat{a}_n$.

W3. Use the results of 299 repetitions of W2 to compute the EDF of $Wald*$. The bootstrap critical value $z_{n,a}*$ is equal to the 1-a quantile of this distribution.

At this point, it is worth remarking that Horowitz (1997, 1999) considers an example in which the bootstrap critical value of the Wald test is computed assuming that U is independent of X and U is normally distributed. With these assumptions, the efficient procedure consists of the following: Estimate q by the maximum likelihood estimator and generate Y values from $Y = [\bar{a}_n(X\bar{b}_n + U*) + 1]^{1/\bar{a}_n}$ where =$\bar{q}_n = (\bar{a}_n, \bar{b}_n')'$ denotes the maximum likelihood estimator and $U*$ is randomly sampled from the normal distribution. This method cannot employed in the present setting because U is, in general, not independent of X, and U is not normally distributed.

Next, consider the Monte Carlo procedure for computing the bootstrap critical value for the LM test.

LM1. Use the estimation data {Y_i, X_i: i = 1, … , n,} to compute the constrained GMM estimator.

LM2. Generate a bootstrap sample by size n by sampling {*Y*, *X*} pairs from the data estimation data with replacement. Compute the constrained GMM estimator of q

from the bootstrap sample. Call the result $\tilde{q}_n^* = (\tilde{a}_n^*, \tilde{b}_n^{*\prime})'$ and $\hat{V}_n^*$. The bootstrap version of the LM statistic requires further study!

(5.3)

LM3. Use the results of 299 repetitions of LM2 to compute the EDF of LM^*. Set $z_{n,a/2}^*$ equal to the 1-a quantile of this distribution.

The true rejection probability of the Wald test of H_0 with a bootstrap critical value is estimated by conducting a Monte Carlo experiment. The experiment consists of repeating the following steps 1000 times:

MC1. Generate an estimation dataset of size n by random sampling from the model with the null hypothesis H_0: $a = a_0$ imposed. Compute the value of the Wald statistic.

MC2. Use the Monte Carlo procedure (W1- W3) for computing the bootstrap critical value, $z_{n,a}^*$.

MC3. Reject H_0 at the nominal a-level if the value of the Wald statistic exceeds $z_{n,a}^*$. The power of the Wald test with the bootstrap critical value is estimated by carrying out the same steps except that $a \neq a_0$ in MC1.

This experiment can also be used to estimate the rejection probability of the Wald test based on the asymptotic critical value $z_{\infty a/2}$. In this case, H_0 is rejected at the nominal a-level if the test statistic exceeds the 1-a quantile of the chi-square distribution with one degree of freedom.

The experiment to estimate the true rejection probability of the LM test based on the bootstrap critical value is similar to the experiment consisting of steps MC1-MC3.

The difference is that in step MC1 the LM statistic is computed instead of the Wald statistic, in step MC2 the Monte Carlo procedure LM1-LM3 is used to compute the bootstrap critical value instead of W1-W3, and in step MC3 the null H_0 is rejected at the nominal a-level if the value of the LM statistic exceeds the bootstrap critical value.

The estimate of the rejection probability under H_0 is computed as R/G where R is the number of rejections of H_0 in G non-deleted estimation samples. A sample is deleted if the minimum value of the objective function essentially exceeds zero, namely, 0.005, when the unconstrained GMM estimate is calculated. In theory, the minimum value of the objective function is zero because the parameters are exactly identified. Nevertheless, in practice, the minimum can be nonzero if the range of the grid for a employed in the grid search is too narrow. For the designs we consider, nonzero values occur very infrequently. In computing the bootstrap critical value for the Wald test, a bootstrap sample is also deleted if the objective function exceeds zero when the unconstrained GMM estimate is calculated.

6. Results of Monte Carlo Experiments

This section reports the results of selected Monte Carlo experiments that illustrate the numerical performance of the Wald and LM tests when they are based on asymptotic critical values and on bootstrap critical values.

Design 1: Truncated Normal, $b_0 = b_1 = 1$, $s = .5$, X uniform [-1, 1]. The empirical rejection probabilities under H_0 are reported in Table 1. They show that the Wald test with asymptotic critical values performs poorly for n = 50 and 100, especially for $a_0 = 0$. The empirical rejection probabilities are within the 99 percent confidence intervals for the nominal rejections levels when n = 200. The distortions are smaller when the test is based on bootstrap critical values. For the bootstrap, the empirical rejection probabilities are within the 99 percent confidence intervals for the nominal rejections levels when n = 100. In the case of the LM test with asymptotic critical values, the differences between the empirical and nominal rejection probabilities are essentially zero at n = 50. The performance of the LM test with bootstrap critical values is work in progress.

Design 2: Gamma, $b_0 = b_1 = 1$, $s = .5$, X perfect uniform [-1, 1] population. The empirical rejection probabilities under H_0 are reported in Table 2. The Wald test with asymptotic critical values performs poorly for n = 50 and 100, but the test does not perform noticeably better with bootstrap critical values, except at the nominal 10 percent level. Again, the LM test with asymptotic critical values has empirical rejection probabilities that are close to the nominal levels.

Design 3: Exponential, $b_0 = 0$, $b_1 = 1$, X uniform [-1, 1]. For the Wald test with asymptotic critical values, Table 3 shows that differences between the empirical and nominal rejection probabilities are essentially zero when n = 100, based on the 99 percent confidence intervals for the nominal rejection probabilities. The Wald test with bootstrap critical values does not perform noticeably better, except at the nominal 10 percent level in some cases. The LM test with asymptotic critical values produces empirical rejection probabilities that are close to the nominal levels for samples of n = 50.

Design 4: Truncated normal, $b_0 = 0$, $b_1 = 1$, $s = .5$, X lognormal. For this design, Table 4 does not report results for the Wald test of H_0: $a = 0$ because the percent of deleted sample is very large, namely more than 50 percent for both NL2SLS and RNL2SLS. For H_0: $a = 1$, the Wald test tends to perform noticeably better with bootstrap critical values than test with asymptotic critical values, starting with n = 50. Again, the

LM test with asymptotic critical values produces empirical rejection probabilities that are close to the nominal levels for samples of n = 50.

Design 5: Truncated normal, $b_0 = 0$, $b_1 = 1$, $s = .5$, X exponential. For the Wald test, Table 5 shows that the differences between the empirical and nominal rejection probabilities are essentially zero when n = 50. Moreover, the Wald test performs noticeably worse with bootstrap critical values than with asymptotic critical values. For the both Design 5 and 6, the differences between the empirical and nominal rejection probabilities are essentially zero at n = 50 for the LM test with asymptotic critical values.

Design 6: Exponential, $b_0 = 0$, $b_1 = 1$, X exponential. With design, there is a striking difference between the performance of the Wald test of H_0: $a = 0$ based on the NL2SLS estimator and on the RNL2SLS estimator. The latter estimator produces much more reasonable results. Nevertheless, neither the asymptotic nor the bootstrap Wald test works for n = 200. For the Wald test of H_0: $a = 1$, the asymptotic and bootstrap critical values give similar empirical rejection probabilities, which tend to be close to the nominal rejection probabilities.

We estimated the powers of the asymptotic and bootstrap Wald tests. The estimated powers are for the tests based on the RNL2SLS estimator. Figure 1 illustrates the empirical powers for Design 3 (Exponential, Uniform) and Design 4 (Truncated Normal, Lognormal). The empirical powers are reported for a 0.01 level test of H_0: $a = 1$ against alternatives 0.8, 0.6, 0.4,0, 0.2 and 0 and for a 0.01 level test of H_0: $a = 0$ against the alternatives a = 0.2, 0.4, 0.6, 0.8 and 1.0. The sample size is n = 50.

In Figure 1, the solid line shows the empirical powers for the tests with asymptotic critical values, and the dashed line shows the empirical powers for the tests with bootstrap critical values. For each design, the asymptotic and bootstrap tests have essentially the same empirical powers. The empirical power functions for Design 3 show that the empirical power of the test of H_0: $a = 1$ against the alternative $a = 0$ is about 0.8 and similarly for the empirical power of the test of H_0: $a = 0$ against $a = 1$. The empirical powers are substantially higher in the case of Design 4, especially for alternatives near the null value of a. For the designs used in this study, the empirical powers show that the Wald test can discriminate between the linear and loglinear models. Our results also show that the design can make a substantial difference as to whether the test can discriminate among local alternatives.

Our plan is to estimate the powers of the LM test.

7. Concluding Comments

For the designs considered in this study, the Wald and LM tests with asymptotic critical values often work reasonably well for samples sizes available in practice. In the case of the Wald test, this finding holds when estimation is based on the rescaled NL2SLS estimator. The differences between the empirical and nominal rejection probabilities under H_0 are small for the Wald test with asymptotic critical values when the sample size is n = 100. The LM test performs better than the Wald test when the null hypothesis is true. The differences between the empirical and nominal rejection probabilities under H_0 are essentially zero for the LM test with n = 50.

The numerical performance of the Wald and LM tests with bootstrap critical values is not noticeably better than with asymptotic critical values for most of the designs in this study. This was not the result we expected. The relatively poor performance of the bootstrap under H_0 may be due a combination of two factors. One is that the bootstrap is based on the sampling of $\{Y, X\}$ pairs, and the other is that the number of bootstrap replications may be too small. Given the semiparametric setting, there is no alternative to sampling $\{Y, X\}$ pairs. We plan to investigate the effect of increasing the number of bootstrap replications.

For the designs in this study, the asymptotic and bootstrap Wald tests have essentially the same empirical powers. Moreover, the Wald test can discriminate between the linear and loglinear models. Our results also show that the design can make a substantial difference as to whether the Wald test can discriminate among local alternatives.

Acknowledgements

The authors thank Joel Horowitz and Whitney Newey for helpful comments.

References

Amemiya, T., 1974. The nonlinear two-stage least-squares estimator. *Journal of Econometrics* 2, 105-110.

Amemiya, T., 1985. *Advanced Econometrics*, Harvard University Press, Cambridge.

Amemiya, T. and Powell, J. L., 1981. A comparison of the Box-Cox maximum likelihood estimator and the non-linear two-stage least-squares estimator. *Journal of Econometrics* 17, 351-381.

Box, G.E.P. and Cox, D.R., 1964. An analysis of transformations. *Journal of the Royal Statistical Society*, Series B 26, 296-311.

Drucker, D. M., 2000. sg1301; On the manipulability of Wald tests in Box-Cox regression models. *Stata Technical Bulletin* 54, 36-42.

Greene, W. H., 2000. *Econometric Analysis, second edition*. Macmillian Publishing Company, New York.

Hall, P. 1992. *The Bootstrap and Edgeworth Expansions*. Springer-Verlag, New York.

Horowitz, J.L., 1997. Bootstrap methods in econometrics. In: Kreps, D.M. and Wallis, K.F. (Eds.), *Advances in Economics and Econometrics: Theory and Applications, Seventh World Congress*, Cambridge, Cambridge University Press, 188-222.

Horowitz, J. L., 1998. *Semiparametric Methods in Econometrics*. Springer-Verlag, New York.

Horowitz, J.L., 1999. The bootstrap in econometrics. In: Heckman, J.J. and Leamer, E.E. (Eds.), *Handbook of Econometrics*, Vol 5. Amsterdam, Elsevier, forthcoming.

Horowitz, J.L. and Savin, N.E., 2000. Empirically relevant critical values for hypothesis tests; a bootstrap approach. *Journal of Econometrics* 95, 375-389.

Khazzoom, D.J., 1989. A note on the application of the non-linear two-stage least-squares estimator to a Box-Cox transformed model. *Journal of Econometrics* 42, 377-379.

Mittlehammer, R. C., Judge. G. C. and Miller, D. J., 2000. *Econometric Foundations*. Cambridge University Press, Cambridge.

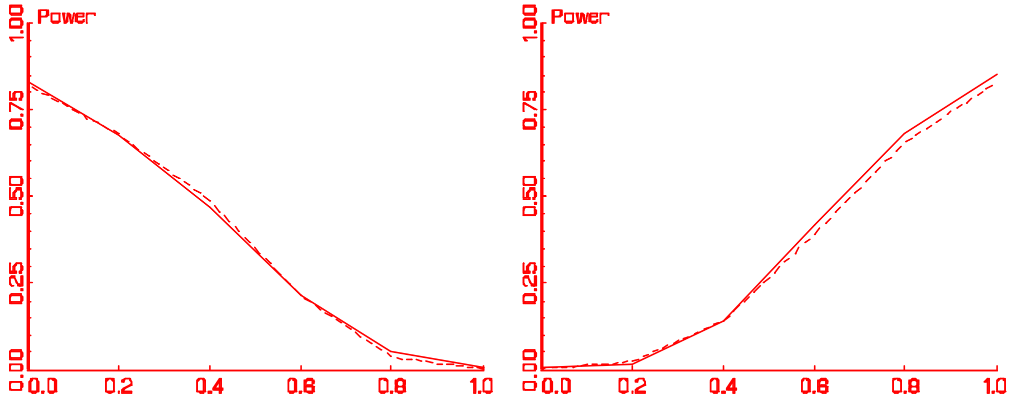
Newey, W.K. and West, K.D., 1987. Hypothesis testing with efficient method of moments estimation. *International Economic Review* 28, 777-787.

Poirier, D. J., 1978. The use of the Box-Cox transformation in limited dependent variable models. *Journal of American Statistical Association* 73, 284-287.

Powell, J. L., 1996. Rescaling methods-of -moments estimation for the Box-Cox regression model. *Economic Letters* 51, 259-265.

Spitzer, J.J., 1984. Variance estimates in models of the Box-Cox transformation: implications for estimation and hypothesis testing. *The Review of Economics and Statistics* 66, 645-652.

Design 3: Exponential, b_0 = 0, b_1 = 1, X uniform [-1, 1]



Design 4: Truncated normal, b_0 = 0, b_1 = 1, s = .5, ***X* lognormal**

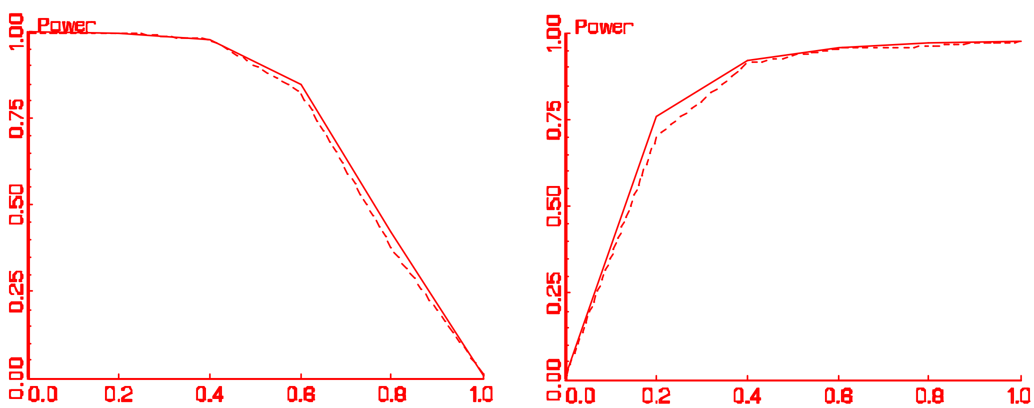


Figure 1. Empirical powers of asymptotic (solid line) and bootstrap (dashed line) Wald tests based on the RNL2SLS estimator, n = 50. The curves on left show the empirical powers of a 0.01 level test of H_0: $a = 1$ against the alternatives 0.8, 0.6, 0.4,0, 0.2 and 0, and the curves on the right show the empirical powers of a 0.01 level test of H_0: $a = 0$ against the alternatives a = 0.2, 0.4, 0.6, 0.8 and 1.0.

Table 1
Empirical Rejection Probabilities (Percent) of Wald and LM Tests

Design: Truncated Normal, $b_0 = b_1 = 1$, $s = .5$, X uniform [-1, 1]										
Critical Value	Hypothesis	1	5	10	1	5	10	1	5	10
		Wald						LM		
		NL2SLS			RNL2SLS			NL2SLS		
n = 50										
Asymptotic	$a = 0$	0.00	0.92	3.28	0.00	1.02	3.37	0.51	5.22	9.72
	$a = 0.5$	0.20	1.32	4.17	0.20	1.43	4.18	1.13	5.12	10.6
	$a = 1$	0.10	0.72	3.90	0.10	1.72	3.81	0.31	4.25	9.43
Bootstrap	$a = 0$	0.21	1.54	4.82	0.31	1.84	5.42			
	$a = 0.5$	0.10	1.12	4.88	0.20	1.33	5.10			
	$a = 1$	0.00	1.23	6.16	0.10	1.55	6.08			
n = 100										
Asymptotic	$a = 0$	0.00	0.91	4.63	0.00	0.90	4.72	1.20	5.22	10.0
	$a = 0.5$	0.10	3.10	6.41	0.01	3.11	6.41			
	$a = 1$	0.20	3.70	6.71	0.20	3.71	6.71	0.80	6.31	12.4
Bootstrap	$a = 0$	0.00	2.72	7.85	0.00	2.61	7.93			
	$a = 0.5$	0.20	3.40	9.71	0.20	3.41	10.0			
	$a = 1$	0.50	3.80	9.81	0.50	4.01	9.92			
n = 200										
Asymptotic	$a = 0$	0.10	2.86	7.97	0.20	3.41	8.83	1.01	5.35	11.4
	$a = 0.5$	0.40	3.50	8.10	0.40	3.50	8.00			
	$a = 1$	0.50	4.00	8.50	0.50	4.00	8.50	1.80	5.80	11.0
Bootstrap	$a = 0$	0.51	5.21	12.0	0.50	5.22	11.8			
	$a = 0.5$	0.40	5.20	11.9	0.40	5.30	11.6			
	$a = 1$	0.60	5.30	10.9	0.60	5.40	10.9			

Notes: The empirical rejection probabilities are computed using 1000 Monte Carlo replications. The 95 percent confidence intervals for the 0.01, 0.05 and 0.10 levels are (0.4, 1.6), (3.6, 6.4) and (8.1, 11.9), respectively; the 99 percent confidence intervals are (0.2, 1.8), (3.2, 6.8) and (7.6, 12.4), respectively.

Table 2

Empirical Rejection Probabilities (Percent) of Wald and LM Tests

Design: Gamma, $b_0 = b_1 = 1$, $s = .5$, X uniform [-1, 1]										
Critical Value	Hypothesis	1	5	10	1	5	10	1	5	10
		Wald						LM		
		NL2SLS			RNL2SLS			NL2SLS		
n = 50										
Asymptotic	$a = 0$									
	$a = 0.5$									
	$a = 1$	0.41	2.27	4.85	0.42	2.28	4.77	1.14	5.41	9.99
Bootstrap	$a = 0$									
	$a = 0.5$									
	$a = 1$	0.41	2.27	6.91	0.62	2.38	6.94	1.04	4.06	9.89
n = 100										
Asymptotic	$a = 0$									
	$a = 0.5$									
	$a = 1$	0.40	2.62	6.05	0.40	2.52	6.05	1.11	4.06	8.78
Bootstrap	$a = 0$									
	$a = 0.5$									
	$a = 1$	0.51	3.73	8.27	0.50	3.83	8.27	0.91	4.44	10.5
n = 200										
Asymptotic	$a = 0$									
	$a = 0.5$									
	$a = 1$									
Bootstrap	$a = 0$									
	$a = 0.5$									
	$a = 1$									

Notes: See Table 1

Table 3

Empirical Rejection Probabilities (Percent) of Wald and LM Tests

Design: Exponential, $b_0 = 0$, $b_1 = 1$, X uniform [-1, 1]										
Critical Value	Hypothesis	1	5	10	1	5	10	1	5	10
		Wald						LM		
		NL2SLS			RNL2SLS			NL2SLS		
n = 50										
Asymptotic	$a = 0$	0.70	3.31	7.11	0.70	3.30	7.11			
	$a = 0.5$									
	$a = 1$	0.90	3.30	7.40	0.90	3.30	7.40	1.00	5.80	10.0
Bootstrap	$a = 0$	0.30	2.71	7.52	0.30	2.70	7.61			
	$a = 0.5$									
	$a = 1$	0.30.	2.60	6.40	0.30.	2.60	6.40			
n = 100										
Asymptotic	$a = 0$	0.30	3.70	7.70	0.30	3.60	7.70			
	$a = 0.5$									
	$a = 1$	0.40	3.60	8.50	0.40	3.60	8.50			
Bootstrap	$a = 0$	0.30	3.60	9.50	0.30	3.50	9.40			
	$a = 0.5$									
	$a = 1$	0.20	3.30	9.30	0.20	3.30	9.30			
n = 200										
Asymptotic	$a = 0$	0.20	4.90	10.5	0.20	4.90	10.4			
	$a = 0.5$									
	$a = 1$	1.20	5.50	11.7	1.20	5.50	11.6			
Bootstrap	$a = 0$	0.50	4.90	11.2	0.50	4.80	10.7			
	$a = 0.5$									
	$a = 1$	0.80	5.20	11.6	0.80	5.20	11.5			

Notes: See Table 1

Table 4
Empirical Rejection Probabilities (Percent) of Wald and LM Tests

Design: Truncated normal, $b_0 = 0$, $b_1 = 1$, $s = .5$, X lognormal										
Critical Value	Hypothesis	1	5	10	1	5	10	1	5	10
		Wald						LM		
		NL2SLS			RNL2SLS			NL2SLS		
n = 50										
Asymptotic	$a = 0$									
	$a = 0.5$									
	$a = 1$	1.93	7.32	12.6	0.90	6.60	12.1	1.40	6.90	12.8
Bootstrap	$a = 0$									
	$a = 0.5$									
	$a = 1$	1.02	4.27	11.0	1.30	5.00	12.1			
n = 100										
Asymptotic	$a = 0$									
	$a = 0.5$									
	$a = 1$	2.39	6.51	12.2	2.61	7.23	13.2			
Bootstrap	$a = 0$									
	$a = 0.5$									
	$a = 1$	1.52	6.84	11.5	1.81	6.83	11.2			
n = 200										
Asymptotic	$a = 0$									
	$a = 0.5$									
	$a = 1$	1.59	5.40	12.7	2.05	6.97	14.8			
Bootstrap	$a = 0$									
	$a = 0.5$									
	$a = 1$	1.58	5.81	10.0	1.43	5.42	8.80			

Notes: See Table 1

Table 5

Empirical Rejection Probabilities (Percent) of Wald and LM Tests

Design: Truncated normal, $b_0 = 0$, $b_1 = 1$, $s = .5$, X exponential										
Critical Value	Hypothesis	1	5	10	1	5	10	1	5	10
		Wald						LM		
		NL2SLS			RNL2SLS			NL2SLS		
n = 50										
Asymptotic	$a = 0$	0.92	4.59	9.99	0.92	4.59	9.99			
	$a = 0.5$									
	$a = 1$	0.80	4.80	10.3	0.90	4.80	10.3	0.70	6.00	11.0
Bootstrap	$a = 0$	0.92	6.01	11.7	0.82	6.42	11.9			
	$a = 0.5$									
	$a = 1$	1.70	6.40	11.8	1.60	6.40	11.9			
n = 100										
Asymptotic	$a = 0$	1.39	5.44	10.8	1.39	5.66	11.0			
	$a = 0.5$									
	$a = 1$	0.90	5.00	11.2	0.90	5.00	11.2			
Bootstrap	$a = 0$	2.45	6.62	12.6	2.56	6.83	12.9			
	$a = 0.5$									
	$a = 1$	1.30	7.20	13.8	1.30	7.20	13.8			
n = 200										
Asymptotic	$a = 0$	1.65	5.82	10.6	1.90	6.20	11.1			
	$a = 0.5$									
	$a = 1$	1.00	5.40	10.1	1.00	5.40	10.1			
Bootstrap	$a = 0$	2.03	6.45	9.88	2.15	6.83	9.98			
	$a = 0.5$									
	$a = 1$	2.00	6.50	12.4	2.00	6.50	12.4			

Notes: See Table 1

Table 6

Empirical Rejection Probabilities (Percent) of Wald and LM Tests

Design: Exponential, $b_0 = 0$, $b_1 = 1$, X exponential										
Critical Value	Hypothesis	1	5	10	1	5	10	1	5	10
		Wald						LM		
		NL2SLS			RNL2SLS			NL2SLS		
n = 50										
Asymptotic	$a = 0$	28.5	31.6	34.6	1.64	6.75	12.0			
	$a = 0.5$									
	$a = 1$	1.70	6.00	10.7	1.80	6.00	10.6	1.60	4.10	9.50
Bootstrap	$a = 0$	0.00	1.00	2.20	0.92	4.09	8.28			
	$a = 0.5$									
	$a = 1$	1.20	4.80	10.9	1.30	4.80	10.8			
n = 100										
Asymptotic	$a = 0$	36.8	40.2	43.9	1.06	7.12	14.1			
	$a = 0.5$									
	$a = 1$	1.90	6.50	11.9	1.80	6.50	11.8			
Bootstrap	$a = 0$	0.00	0.30	1.00	0.74	3.83	6.38			
	$a = 0.5$									
	$a = 1$	1.90	6.60	11.4	1.90	6.40	11.5			
n = 200										
Asymptotic	$a = 0$	37.8	41.4	44.4	2.94	9.62	15.1			
	$a = 0.5$									
	$a = 1$	1.40	5.50	10.8	1.40	5.70	11.1			
Bootstrap	$a = 0$	0.00	0.10	0.20	0.94	7.05	16.2			
	$a = 0.5$									
	$a = 1$	1.20	5.40	10.5	1.30	5.4	10.6			

Notes: See Table 1