A unified approach to testing for stationarity and unit roots

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Abstract

Lagrange multiplier tests against nonstationary unobserved components such as stochastic trends and seasonals are based on statistics which, under the null hypothesis, have asymptotic distributions belonging to the class of generalised Cramér-von Mises distributions. Conversely, unit root tests can be formulated, again using the Lagrange multiplier principle, so as to yield test statistics which also have Cramer-von Mises distributions under the null hypothesis. These ideas may be extended to multivariate models and to models with structural breaks thereby providing a simple unified approach to testing in nonstationary time series.

KEYWORDS: Cramér-von Mises distribution, Co-integration, Lagrange multiplier test, Locally best invariant test, Seasonality, Stochastic trend.

JEL classification: C22, C32

1. Introduction

In a unit root test, the null hypothesis that a process contains a unit root while the alternative is that it is stationary. Stationarity tests operate in the opposite direction. The null hypothesis is that the series is stationary, while the alternative is that a nonstationary component is present; see Nyblom and Mäkeläinen (1983) and Kwiatkowski, Phillips, Schmidt and Shin (1992). Used in the context of testing the validity of a pre-specified co-integrating vector, the null hypothesis is that the co-integrating relationship is true. The asymptotic distribution of the stationarity test statistic under the null hypothesis is the Cramér-von Mises distribution. When a time trend is present the distribution is different but can still be regarded as belonging to the same family. Furthermore the test statistic against the presence of a multivariate random walk and the seasonality test of Canova and Hansen (1995) both have asymptotic distributions under the null hypothesis which belong to a class of generalised Cramér-von Mises distributions, indexed by a degree of freedom parameter.

The most widely used unit root test is the (augmented) Dickey-Fuller (ADF) test; see Fuller (1996, ch 10) and the references therein. However, the autoregressive model means that the roles of the constant and time trend are different under the null and alternative hypotheses. This problem may be avoided by working with models set up in terms of components; see, for example, the discussion in Maddala and Kim (1998, pp 37-9) and the papers by, amongst others, Bhargava (1986), Nabeya and Tanaka (1990) and Schmidt and Phillips (1992). As with stationarity tests, the components framework leads naturally to unit root tests which derive from the Lagrange multiplier principle rather than being Wald tests. What has apparently not been noticed is that it is possible to formulate the LM-type unit root tests in such a way that under the null hypothesis the test statistics have asymptotic distributions belonging to the Cramér-von Mises family. This extends to multivariate and seasonality tests. Thus unit root and stationarity tests display an appealing symmetry - or perhaps asymmetry in that the critical values for the unit root tests are in the lower tail of the Cramér-von Mises distributions while those for the stationarity tests are in the upper tails.

The plan of the paper is as follows. Section 2 reviews the theory of stationarity tests. The relative merits of dealing with serial correlation by the nonparametric approach of Kwiatkowski, Phillips, Schmidt and Shin (1992) as opposed to a parametric approach based on estimating an unobserved components models are discussed. The extensions to testing against nonstationary seasonal components and stochastic slopes are then reviewed.

Section 3 shows how unit root test can be set up so that the test statistics have asymptotic distribution which belong to the Cramér-von Mises family under the null hypothesis. For general unobserved components models the test statistics can be constructed very easily using standardized innovations - one-step ahead prediction errors - produced by the Kalman filter. We will refer to such parametric tests as *unobserved components unit root* tests.

Section 4 extends the ideas of section 3 to multivariate models. In particular, a multivariate unit root test with a Lagrange multiplier interpretation and a Cramér-von Mises distribution is suggested. Section 5 looks at seasonal unit root tests, suggesting an LM alternative to the procedure of Hylleberg, Engle, Granger and Yoo (1992). A test of the null hypothesis that there is a unit root in the slope of a trend is derived in section 6.

Section 7 follows Busetti and Harvey (2001) in showing how the stationarity tests are affected by structural breaks in a series which are modelled by dummy variables. Although the form of the test statistics is unchanged, their asymptotic distributions are altered. However, the additive properties of the Cramér-von Mises distribution suggest a simplified test which is much easier to implement. The effect on LM type unit root tests is then examined. These remain the same unless there are breaks in the slope in which case a modification along the lines proposed for stationarity tests leads to simplified statistics with Cramér-von Mises distributions under the null hypothesis. Similar results hold for seasonality tests when breaks in the seasonal pattern are modelled by dummy variables. All of these tests extend to multivariate models.

2. Stationarity tests

This section reviews the literature on testing against the presence of nonstationary unobserved components. The leading case, testing against a random walk in an otherwise stationary series, is sometimes called a stationarity test. In adopting this terminology more widely it must be realized that the model may contain other nonstationary components, such as seasonals, which remain present under the null hypothesis.

2.1. Testing against the presence of a random walk component

Consider a univariate unobserved components model consisting of a random walk plus noise for a set of observations, y_t :

$$y_t = \mu_t + \varepsilon_t, \qquad \mu_t = \mu_{t-1} + \eta_t, \quad t = 1, ..., T, \tag{2.1}$$

where the $\eta_t's$ and $\varepsilon_t's$ are mutually and serially uncorrelated Gaussian disturbances with variances σ_η^2 and σ_ε^2 respectively. When $\sigma_\eta^2 = 0$ the random walk becomes a constant level. Nyblom and Mäkeläinen (1983) showed that the locally best invariant test (LBI) test of the null hypothesis that $\sigma_\eta^2 = 0$, against the alternative that $\sigma_\eta^2 > 0$, can be formulated as

$$\eta = T^{-2} \sum_{i=1}^{T} \left[\sum_{t=1}^{i} e_t \right]^2 / s^2 > c, \tag{2.2}$$

where $e_t = y_t - \overline{y}$, $s^2 = T^{-1}\sum_{t=1}^{T}(y_t - \overline{y})^2$ and c is a critical value. In fact, one initially obtains a form of the statistic with the summations running in reverse, that is from $t = i$ to T, but it is easily seen that the two statistics are identical. The test can also be interpreted as a one-sided Lagrange multiplier (LM) test. Because numerical optimisation is needed to estimate (2.1), implementing likelihood ratio and Wald tests is less straightforward; see Kuo (1999).

The asymptotic distribution of the statistic under the null hypothesis is found by first observing that the partial sum of deviations from the mean converges weakly to a standard Brownian bridge, that is

$$\sigma^{-1}T^{-\frac{1}{2}}\sum_{s=1}^{[Tr]} e_s \Rightarrow B(r), \qquad r \in [0,1] \tag{2.3}$$

where $[Tr]$ is the largest integer less than or equal to Tr and $B(r) = W(r) - rW(1)$, with $W(\cdot)$ being a standard Wiener process or Brownian motion. Hence

$$\eta \Rightarrow \int_0^1 B(r)^2 dr \tag{2.4}$$

since $s^2 \xrightarrow{p} \sigma^2$. This is the Cramér-von Mises distribution, denoted as CvM. It is sufficient for the observations to be independent and identically distributed (with finite variance) to yield this asymptotic distribution.

If a linear time trend is included in (2.1) so that

$$y_t = \mu_t + \beta t + \varepsilon_t, \qquad t = 1, ..., T, \tag{2.5}$$

the test statistic, η_2, is as in (2.2) except that it is formed from the OLS residuals from a regression on a constant and time. The partial sum of residuals weakly converges to a second level Brownian bridge, denoted $B_2(\cdot)$. Then

$$\eta_2 \Rightarrow \int_0^1 B_2(r)^2 dr \ . \tag{2.6}$$

This is a second level Cramér-von Mises distribution, denoted $CvM_{2.}$ In the case of any ambiguity the distribution in (2.4) will be referred to as CvM_1.

2.2. Serial correlation

Now suppose that the model is extended so that ε_t is any indeterministic stationary process. In this case the asymptotic distribution of the η test statistic remains the same if s^2 is replaced by a consistent estimator of the long-run variance (the spectrum at frequency zero)

$$\sigma_L^2 = \lim_{T\rightarrow\infty} T^{-1}E\left[\left(\sum_{t=1}^{T}\varepsilon_t\right)^2\right] = \sum_{\tau=-\infty}^{\infty}\gamma(\tau) \tag{2.7}$$

where $\gamma(\tau)$ is the autocovariance of ε_t at lag τ. Kwiatkowski et al (1992) - KPSS - construct such an estimator nonparametrically as

$$s_L^2(\ell) = T^{-1}\sum_{t=1}^{T}e_t^2 + 2T^{-1}\sum_{\tau=1}^{\ell}w(\tau,\ell)\sum_{t=\tau+1}^{T}e_te_{t-\tau} = \hat{\gamma}(0) + 2\sum_{\tau=1}^{\ell}w(\tau,\ell)\hat{\gamma}(\tau) \tag{2.8}$$

where $w(\tau,\ell)$ is a weighting function, such as $w(\tau,\ell) = 1-\tau/(\ell+1)$, $\tau = 1,...,\ell$. In what follows this statistic will be referred to as $KPSS(\ell)$. Other weighting functions, such as the Parzen or Tukey windows, may be used.

Leybourne and McCabe (1994) attack the problem of serial correlation by introducing lagged dependent variables into the model. The test statistic obtained after removing the effect of the lagged dependent variables is then of the same form as (2.2). The practical implication, as demonstrated in their Monte Carlo results, is a gain in power. However, more calculation is involved since the coefficients of the lagged dependent variables are estimated under the alternative hypothesis and this requires numerical optimization.

Since we are testing for the presence of an unobserved component it seems natural to work with structural time series models. If the process generating the stationary part of the model were known, the LBI test for the presence of a random walk component could be constructed. Harvey and Streibel (1997) derive such a test and show how it is formed from a set of 'smoothing errors'. A general algorithm for calculating these statistics is the Kalman filter and associated smoother. The smoothing errors are, in general, serially correlated but the form of this serial correlation may be deduced from the specification of the model. Hence a (parametric) estimator of the long-run variance may be constructed and used to form a statistic which has a Cramér-von Mises distribution, asymptotically, under the null hypothesis. An alternative possibility is to use the standardized one-step ahead prediction errors (innovations), calculated assuming that μ_0 is fixed[1]. No correction is then needed and, although the test is not strictly LBI, its asymptotic distribution is the same and the evidence presented in Harvey and Streibel

[1] This requires estimating μ_0 by smoothing. Another possibility is to reverse the order of the observations and to calculate innovations starting from the filtered estmator of μ_T. This avoids smoothing.

(1997) suggests that, in small samples, it is more reliable in terms of size. As in the Leybourne-McCabe test, the nuisance parameters need to be estimated and this is best done under the alternative hypothesis. This has the compensating advantage that since there will often be some doubt about a suitable model specification, estimation of the unrestricted model affords the opportunity to check its suitability by the usual diagnostics and goodness of fit tests. Once the nuisance parameters have been estimated, the test statistic is calculated from the innovations or the smoothing errors with σ_η^2 set to zero.

An advantage of the unobserved components approach is that it can easily accommodate the kind of evolving seasonal pattern that is often a feature of monthly or quarterly data. Within a structural time series model framework, seasonality is modelled by a nonstationary component, γ_t, which has the property that

$$S(L)\gamma_t \sim MA(s-2) \tag{2.9}$$

where $S(L) = 1 + L + \ldots + L^{s-1}$ is the seasonal summation operator and s is the number of seasons; see Harvey (1989, ch 2) and the next sub-section. The inclusion of such a component in the model has no effect on testing procedures in that the innovations can be used exactly as before.

2.3. Testing against nonstationary seasonality

Consider a Gaussian model with a trigonometric seasonal component:

$$y_t = \mu + \gamma_t + \varepsilon_t, \qquad t = 1, ..., T \tag{2.10}$$

where μ is a constant and

$$\gamma_t = \textstyle\sum_{j=1}^{[s/2]} \gamma_{j,t} , \tag{2.11}$$

where each $\gamma_{j,t}$ is generated by

$$\begin{bmatrix} \gamma_{j,t} \\ \gamma_{j,t}^* \end{bmatrix} = \begin{bmatrix} \cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j \end{bmatrix} \begin{bmatrix} \gamma_{j,t-1} \\ \gamma_{j,t-1}^* \end{bmatrix} + \begin{bmatrix} \omega_{j,t} \\ \omega_{j,t}^* \end{bmatrix}, \qquad \begin{matrix} j = 1, \ldots, [s/2], \\ t = 1, \ldots, T, \end{matrix} \tag{2.12}$$

where $\lambda_j = 2\pi j/s$ is frequency, in radians, and $\omega_{j,t}$ and $\omega_{j,t}^*$ are two mutually uncorrelated white noise disturbances with zero means and common variance σ_j^2. For s even $[s/2] = s/2$, while for s odd, $[s/2] = (s-1)/2$. For s even, the component at $j = s/2$ collapses to

$$\gamma_{j,t} = \gamma_{j,t-1}\cos\lambda_j + \omega_{j,t}, \qquad j = s/2. \tag{2.13}$$

If ε_t is white noise, the LBI test against the presence of a stochastic trigonometric component at any one of the seasonal frequencies, λ_j, apart from the one at π, is

$$\omega_j = 2T^{-2}s^{-2}\sum_{i=1}^{T}\left[\left(\sum_{t=1}^{i} e_t \cos\lambda_j t\right)^2 + \left(\sum_{t=1}^{i} e_t \sin\lambda_j t\right)^2\right], \qquad j = 1, ..., [s/2], \tag{2.14}$$

where s^2 is the sample variance of the OLS residuals from a regression on sines and cosines. Canova and Hansen (1995) show that the asymptotic distribution of this statistic is (generalized) Cramér-von Mises with two degrees of freedom[2], that is $CvM_1(2)$. The component at π gives rise to a test statistic which has only one degree of freedom. A joint test against the presence of stochastic trigonometric components at all seasonal frequencies is based on a statistic obtained by summing the individual test statistics.[3] This statistic has an asymptotic distribution which is $CvM_1(s-1)$. If desired it can be combined with a test against a random walk to give a test statistic which is $CvM_1(s)$ when both level and seasonal are deterministic.

Canova and Hansen show how the above tests can be generalized to handle serial correlation by making a correction similar to that in KPSS, the difference being that the correction now involves the spectrum at seasonal frequencies rather than at zero. If the model contains a stochastic trend, then the test must be carried out on differenced observations. A parametric test may be carried out by fitting an unobserved components model. If there is a trend it may be a deterministic trend, a random walk, with or without drift, or a trend with a stochastic slope, as defined in the next sub-section. The examples given in Busetti and Harvey (2000) indicate that the parametric test is far more effective.

2.4. Testing against a stochastic slope

Generalizing the trend in (2.1) to include a stochastic slope gives

$$\begin{array}{rcll} \mu_t & = & \mu_{t-1} + \beta_{t-1} + \eta_t, & \eta_t \sim NID(0, \sigma_\eta^2), \\ \beta_t & = & \beta_{t-1} + \zeta_t, & \zeta_t \sim NID(0, \sigma_\zeta^2), \end{array} \tag{2.15}$$

where $NID(0, \sigma_\eta^2)$ denotes normally and independently distributed and the level and slope disturbances, η_t and ζ_t, respectively, are mutually independent. If σ_η^2

[2] Actually Canova and Hansen derive the above statistic from a slightly different form of the stochastic cycle model in which the coefficients of a sine-cosine wave are taken to be random walks. However, it is not difficult to show that the model as defined above leads to the same test statistic

[3] This is the LM test if $\sigma_j^2 = \sigma_\omega^2$ for all j except $j = s/2$ when $\sigma_{s/2}^2 = \sigma_\omega^2/2$.

is assumed to be zero the trend, μ_t, is an integrated random walk, or 'smooth trend'. Nyblom and Harvey (2001) derive the asymptotic distribution of the LBI test of $H_0 : \sigma_\zeta^2 = 0$ against $H_1 : \sigma_\zeta^2 > 0$. However, a Monte Carlo study of the test seems to show that it offers little gain in power over η_2. Whichever test is chosen, nuisance parameters should be estimated by fitting the smooth trend model.

If $\sigma_\eta^2 > 0$, then, when $\sigma_\zeta^2 = 0$ the trend reduces to a random walk plus drift. Differencing yields

$$\Delta y_t = \beta_{t-1} + \eta_t + \Delta\varepsilon_t, \qquad t = 2, ..., T \tag{2.16}$$

with $\eta_t + \Delta\varepsilon_t$ being invertible. The test statistic, denoted ζ, for testing whether β_{t-1} is a random walk can be constructed as in sub-section 2.2, but its asymptotic distribution is CvM_1 rather than CvM_2.

2.5. The family of Cramér-von Mises distributions

The asymptotic distributions of the various test statistics described above suggest a family of Cramér-von Mises distributions, denoted $CvM_{p+1}(k)$, dependent on degrees of freedom, k, and whether or not a constant ($p = 0$) or a time trend ($p = 1$) is fitted. (Recall that when $p = 0$, the $p+1$ subscript is often dropped). The distribution when no deterministic component is fitted ($p = -1$) is $CvM_0(k)$. An example is the general test for seasonality proposed by Busetti and Harvey (2000), in which no seasonal dummies are fitted when the Canova-Hansen statistic is formed, and the asymptotic distribution is $CvM_0(s-1)$. MacNeill (1978, p431) considers fitting polynomials of degree p and tabulates $CvM_{p+1}(1)$ for $p = -1, 0, 1, .., 5$. If a stationarity test is applied to $d-th$ differences, as with the test for a stochastic slope, then the asymptotic distribution of the test statistic, ζ, is $CvM_{p-d+1}(1)$.

The Cramér-von Mises distribution with $p = 0$ and k degrees of freedom may be expanded as

$$CvM(k) = \sum_{j=1}^{\infty}(\pi j)^{-2}\chi_j^2(k), \tag{2.17}$$

There are similar series expansion representation for other members of the family. In particular for $CvM_0(k)$ the weights are $\pi^{-2}(j-1/2)^{-2}$, while for $CvM_2(k)$, the weights are obtained by changing $(\pi j)^{-2}$ to λ_j^{-2}, where $\lambda_{2j-1} = 2j\pi$ and λ_{2j} is the root of $\tan(\lambda/2)=\lambda/2$ on $(2j\pi, 2(j+1)\pi)$, $j = 1, 2, ...$ An important corollary is that, because of the additive property of chi-square distributions, the sum of two independent random variables with distributions $CvM(k_1)$ and $CvM(k_2)$ is $CvM(k_1 + k_2)$.

It follows from the series expansion in (2.17) that $E[CvM(k)] = k/6$ and $Var[CvM(k)] = k/45$. As $k \rightarrow \infty$, each chi-square distribution may be approximated by a normal and so $CvM(k)$ may also be approximated by a normal. Hence the 5% critical value for large k may be approximated by $k/6+1.645\sqrt{k/45}$. For $k = 4$, this yields 1.159 as opposed to the value of 1.237, while for $k = 11$, the approximate value is 2.646 as against 2.739.

The test for a stochastic slope in a smooth trend model, introduced in the previous sub-section, suggests a further generalization of the family of Cramér-von Mises distributions. This generalization will not be pursued here.

3. Unit root tests

3.1. Lagrange multiplier tests

The Dickey-Fuller test is based on the model

$$y_t = \alpha + \beta t + \phi y_{t-1} + \xi_t, \qquad \xi_t \sim NID(0, \sigma^2), \qquad t = 1, ..., T \tag{3.1}$$

with variations in which the trend and both the constant and the trend are omitted. The null is that ϕ is unity, so the model is nonstationary, while the alternative is that it is less than unity, so the model is (trend) stationary. If the model is reformulated with Δy_t as the dependent variable, the parameter associated with y_{t-1}, and denoted here as ρ, is equal to $\phi - 1$ and hence is zero under the null hypothesis. The test statistic is based on the regression coefficient of the lagged dependent variable or its 't-statistic'. Lagged differences can be added to the right hand side without affecting the asymptotic distribution of the estimator of ρ.

Formulating the unit root test in an autoregressive framework is computationally convenient. However, as Schmidt and Phillips (1992, p 258) observe, the parameterizations of (3.1) are "...not convenient..." because "...they handle level and trend in a clumsy and potentially confusing way." Specifically the meanings of α and β differ under the null and alternative hypotheses. These difficulties can be avoided by following Bhargava (1986), Nabeya and Tanaka (1990) and Schmidt and Phillips (1992) and setting up the unit root test of $H_0 : \phi = 1$ against $H_0 : \phi < 1$ within the components framework

$$y_t = \alpha + \beta t + \mu_t, \qquad \mu_t = \phi \mu_{t-1} + \eta_t, \quad t = 1, ..., T, \tag{3.2}$$

The interpretation of α and β is now the same under both the null and alternative hypotheses.

Schmidt and Phillips (1992) show that LM tests of the unit root hypothesis are based on the residuals obtained by estimating α and β under the null hypothesis. Since

$$\Delta y_t = \beta + \eta_t, \quad t = 2, ..., T \tag{3.3}$$

under the null hypothesis, these residuals are defined by

$$\widetilde{\mu}_t = y_t - \widetilde{\alpha}_0 - \widetilde{\beta} t, \qquad t = 1, ..., T$$

where $\widetilde{\beta} = \overline{\Delta y} = \sum \Delta y_t/(T-1) = (y_T - y_1)/(T-1)$ and $\widetilde{\alpha}_0 = y_1 - \widetilde{\beta}$, where $\alpha_0 = \alpha + \mu_0$. Note that $\widetilde{\mu}_1 = 0$, while $\widetilde{\mu}_T = 0$ provided a slope, β, is estimated. Schmidt and Phillips (1992) formulate their test in terms of a regression analogous to the one used in the Dickey-Fuller test, with y_{t-1} replaced by $\widetilde{\mu}_{t-1}$ and a constant but no time trend included. The tests are based on the regression coefficient of $\widetilde{\mu}_{t-1}$ or its 't-statistic'. A variant of the test, studied further in Schmidt and Lee (1991), excludes the constant.

Now consider the test with critical region

$$T^{-1} \sum_{t=1}^{T} \widetilde{\mu}_t^2 / \sum_{t=1}^{T} (\widetilde{\mu}_t - \widetilde{\mu}_{t-1})^2 = \zeta < c \tag{3.4}$$

where $\widetilde{\mu}_0$ is taken to be zero. This corresponds to the N_2 test suggested by Bhargava (1986) except insofar as his test statistic, being of the von Neumann ratio form, is equal to $1/T\zeta$. It is a transformation of the Schmidt-Phillips test statistic given as T times the coefficient obtained by regressing $\Delta\widetilde{\mu}_t$ on $\widetilde{\mu}_{t-1}$ without a constant term; it follows from Schmidt and Phillips (1992)and the appendix that this statistic is equal to $-1/2\zeta$. The ζ statistic is the same as R_4 in Nabeya and Tanaka (1990), who show that the test is locally best invariant and unbiased (LBIU). If it is written in first differences it becomes

$$\zeta = T^{-1} \sum_{i=1}^{T} \left[\sum_{t=1}^{i} \Delta\widetilde{\mu}_t \right]^2 / \sum_{t=1}^{T} (\Delta\widetilde{\mu}_t)^2 \tag{3.5}$$

This is of the same form as the η test statistic, (2.2), except that it applies to observations in first differences. Provided the slope is estimated so that $\Delta\widetilde{\mu}_t = \Delta y_t - \overline{\Delta y}$ for $t = 2, .., T$, it is immediately apparent that the statistic has a CvM_1 distribution under the null hypothesis. However, while the value of the stationarity statistic, η, increases under the alternative, the value of ζ decreases as it is $T\zeta$ which has a limiting distribution under the alternative; compare Schmidt and Phillips (1992, p 267). Thus the appropriate critical values are those in the lower (left-hand) tail of the CvM distribution.

If there is no time trend in the model[4], $\overline{\Delta y}$ is omitted and the asymptotic distribution of the statistic is CvM_0. In this case it is useful to label the statistic ζ_1, and to denote the time trend statistic as ζ_2 when there is any ambiguity. If there is neither constant nor time trend, so that the statistic, ζ_0, is constructed by setting $\widetilde{\mu}_t = y_t$ for all $t = 0, 1, ..., T$ the asymptotic distribution is again CvM_0 (although a non-zero initial value has more effect on the small sample distribution). From Tanaka (1996, table 9.1), the critical values at the 5% and 1% levels of significance are 0.0565 and 0.0345 if no time trend is included and 0.0366 and 0.025 if one is included[5].

The modified statistic

$$T^{-1}\sum(\widetilde{\mu}_t - \overline{\widetilde{\mu}})^2/\sum(\widetilde{\mu}_t - \widetilde{\mu}_{t-1})^2, \qquad (3.6)$$

where $\overline{\widetilde{\mu}}$ is the mean of the $\widetilde{\mu}_t's$, has an asymptotic distribution in which the $B(r)$ in (2.4) is replaced by a de-meaned Brownian bridge. It is a transformation of the test statistic in Schmidt and Phillips (1992) and the R_2 statistic in Bhargava (1986). It corresponds directly to R_3 in Nabeya and Tanaka(1990) and from their table 1 the 5% critical values are 0.036 if no time trend is included and 0.027 if one is included. Schmidt and Lee (1991) compare the tests based on (3.5) and (3.6) using Monte Carlo simulations and seem to come down in favour of (3.6) though the evidence is by no means clear-cut. Nabeya and Tanaka (1990), using an analysis based on limiting powers find that there is no dominance of one test over the other for the time trend model considered by Schmidt and Lee (1991). Furthermore, if a time trend is not present, then ζ is better. Further discussion can be found in Tanaka (1996, p348), where ζ is labelled R_2.

The distribution theory surrounding ζ can be generalised by letting the deterministic part of (3.2) be a $p - th$ order polynomial. The residuals in (3.5) are then obtained by regressing Δy_t on a polynomial of order $p-1$ with the result that the test statistic ζ_{p+1}, is asymptotically CvM_p under the null hypothesis.

The right hand tail of ζ can be used to test against explosive processes, that is $\phi > 1$; see Bhargava (1986) and Nabeya and Tanaka (1990). However, another interpretation of the alternative, which fits more nicely into the stochastic trends framework, is that the test is against a stochastic slope. In other words it is the test motivated by (2.16): hence the ζ notation.

[4]The test is no longer the LM test when there is no time trend; see the discussion in Tanaka (1996, ch 9).

[5]Nabeya and Tanaka (1990, table 4) give finite sample critical values. For example with a time trend the 5% critical values for $T = 25$ and 50 are 0.042 and 0.039 respectively.

3.2. Serial correlation and unobserved components

Nabeya and Tanaka(1990) consider methods of adjusting the statistics (3.5) and (3.6) so that the same asymptotic distribution is obtained under the null hypothesis when η_t is serially correlated. They suggest using a nonparametric estimator of the long-run variance, constructed in a similar way to (2.8) with $\Delta\widetilde{\mu}_t$ replacing e_t. This corresponds to the KPSS statistic computed from first differences. However, under the alternative the spectrum of first differences is zero at the origin. Schmidt and Phillips (1992, p267) make a similar proposal but argue that a consistent test requires the use of residuals obtained (under the alternative hypothesis) from a Dickey-Fuller regression based on (3.1). Another option would be to base a test on the coefficient of $\widetilde{\mu}_{t-1}$ from an augmented Dickey-Fuller regression as in Oya and Toda (1998)

If a fully parameterized UC model is set up, an LM-type test may be carried out by estimating the model under the null hypothesis and then forming a test statistic from the standardized innovations, $\widetilde{\nu}_t$. These are calculated starting with the smoothed estimator of μ_0 so they run from $t = 1$ to T. Assuming the innovations have been standardized so as to have unit variance, the unobserved components unit root test statistic is simply

$$\zeta = T^{-2} \sum_{i=1}^{T} \left[\sum_{t=1}^{i} \widetilde{\nu}_t \right]^2 \tag{3.7}$$

Alternatively smoothing may be avoided by reversing the order of the observations and calculating (a different set of) innovations starting from the filtered estimator of μ_T.

The case for a parametric UC approach can be illustrated simply by adding white noise to (3.2) to give

$$y_t = \alpha + \beta t + \mu_t + \varepsilon_t, \qquad t = 1, ..., T. \tag{3.8}$$

This model is easily estimated when $\phi = 1$ and so forming the test statistic from the innovations, as in (3.7), is straightforward. (Note that if σ_ε^2 is zero, so that the model reduces to (3.2), then $\widetilde{\nu}_1 = 0$). Applying the Dickey-Fuller test when the data are best approximated by (3.8) is likely to result in too many rejections under the null hypothesis if the ratio of σ_η^2 to σ_ε^2 is low. The reduced form is an ARIMA(0,1,1) model with MA parameter close to minus one and the poor performance of the augmented Dickey-Fuller test is well documented in this situation; see, for example, Pantula (1991). Nonparametric corrections based on the estimation of the long-run variance, as in Schmidt and Phillips (1992), are also likely to be poor for this kind of model for the reasons given in Perron and Mallet (1996).

Stochastic Volatility- The discrete time Gaussian SV model may be written as

$$r_t = \sigma_t \varepsilon_t = \sigma \varepsilon_t e^{0.5h_t}, \qquad \varepsilon_t \sim NID(0,1), \quad t = 1, ..., T,$$

where r_t is a return on an exchange rate or stock price, σ is a scale parameter and h_t is a stationary first-order autoregressive process

$$h_t = \phi h_{t-1} + \eta_t, \qquad \eta_t \sim NID(0, \sigma_\eta^2) \tag{3.9}$$

Squaring the observations and taking logarithms gives

$$y_t = \log r_t^2 = \log \sigma^2 + h_t + \log \varepsilon_t^2, \quad t = 1, ..., T. \tag{3.10}$$

Ignoring the time trend, the model is as in (3.8), except that $\log \varepsilon_t^2$ is far from being Gaussian, being heavily skewed with a long tail. However, this makes no difference to the asymptotic distribution of the test statistics we are about to consider.

In the application in Harvey *et al.* (1994), the r_t is the difference of 946 logged daily exchange rates of the dollar against another currency starting on 1st November 1981; the data are provided with the STAMP package Koopman *et al* (2000). Various tests were applied to the observations transformed with a modification made to $\log r_t^2$ to avoid distortion from inliers; see Fuller (1996, p 496). The same transformation was used when the estimates of the ϕ parameters were obtained by quasi-ML using STAMP.

The results of the η test are shown in table 1. All the values apart from the Deutschmark are significant[6] at the 1% level indicating the presence of a random walk or, perhaps, a very persistent AR(1) component in volatility. Note the reduction in power if a KPSS correction is (unnecessarily) made. Higher lag length leads to even smaller statistics. For example, $KPSS(25)$ for the Pound is 0.515. The unit root test statistics are also shown in the table. None of the ζ statistics rejects at any conventional level of significance. Indeed their values are comfortably located near the median of the null hypothesis asymptotic distribution. The fact that the ADF $t-$statistics (with constant included) all lie way beyond the 1% asymptotic critical value of -3.42 is a reflection of the fact that the autoregressive approximation is

[6] The 1%, 5% and 10% upper tail critical values for CvM_1 are 0.743, 0.461 and 0.347 respectively.

very poor because σ_η^2 is dominated by the variance of $\log \varepsilon_t^2$. However, if the lag length is increased to 25, the ADF statistic for the Pound is -3.37 and so just fails to reject. The poor autoregressive approximation has a similar effect on the Oya-Toda version of the LM type test.[7]

Table 1 Tests of stochastic volatility of daily exchange rates

Currency	η	$KPSS(9)$	ζ	$ADF(9)$	$\widetilde{\phi}$
Pound	1.319	0.853	.228	-6.43	.988
DM	0.423	0.256	.371	-7.50	.967
Yen	5.122	2.999	.439	-7.63	.998
Swiss Fr	0.774	0.465	.466	-7.44	.980

The model in (3.8) may be generalised by including other components such as seasonals and cycles. Such models are easily estimated with ϕ set to one. The η statistic is computed from the innovations obtained from the Kalman filter by setting σ_η^2 to zero. Its aim is to determine whether a restriction should be placed on the model, while the ζ test is to find out if it should be more general.

Quarterly consumption - Harvey and Scott (1994) showed that a model consisting of a random walk with drift and a stochastic seasonal component gives a good fit to quarterly UK non-durable consumption.[8] The ζ statistic calculated from the innovations from this model is 0.165. This is well away from the lower tail 10% critical value for the CvM_1 distribution which is 0.025 and so we cannot reject the hypothesis that the stochastic trend component is a random walk against the alternative that it is a stationary AR(1) process. The same statistic[9] can be used to test the null hypothesis that the slope, β, is constant against the alternative that it is a random walk; see sub-section 2.4. It is the upper tail of the CvM_1 distribution which is now relevant, but the 10% point is 0.347 so again the null is clearly not rejected.

[7] Having subtracted the first observation, the ADF(9) t-statistic is found to be -6.44, while the estimate of ϕ is 0.501.

[8] The data are given in the STAMP package. As in Harvey and Scott the sample period is 57q3 to 92q2. The estimates of the level and seasonal variances are found to be 8.908×10^{-5} and 1.012×10^{-6}, which differ slightly from those reported in Harvey and Scott due to small revisions in the data.

[9] The nuisance parameters are normally estimated under the alternative for a 'stationarity' test. In this context it makes little difference since the seasonal variance is not sensitive to the specification of the trend.

3.3. Seasonal unit root tests

The test of Hylleberg et al. (1990) - HEGY- is related to CH in that it is testing the null of a nonstationary seasonal against the alternative of a stationary seasonal; thus it parallels the relationship between (augmented) Dickey-Fuller test and KPSS. Rodrigues (2000) sets up LM type tests within the Schmidt-Phillips, or rather Schmidt-Lee, framework for the autoregressive model used by HEGY.

The UC seasonal unit root test can be set up by introducing a damping factor into (2.12) so that each trigonometric term in the seasonal component is modelled by

$$\begin{bmatrix} \gamma_{j,t} \\ \gamma^*_{j,t} \end{bmatrix} = \phi_j \begin{bmatrix} \cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j \end{bmatrix} \begin{bmatrix} \gamma_{j,t-1} \\ \gamma^*_{j,t-1} \end{bmatrix} + \begin{bmatrix} \omega_{j,t} \\ \omega^*_{j,t} \end{bmatrix}, \qquad \begin{matrix} j = 1, \ldots, [s/2], \\ t = 1, \ldots, T. \end{matrix} \tag{3.11}$$

For s even

$$\gamma_{j,t} = \phi_j \gamma_{j,t-1} \cos\lambda_j + \omega_{j,t}, \qquad j = s/2. \tag{3.12}$$

The seasonal component, obtained by summing the $\gamma'_{j,t}s$ is then embedded in a general UC model which contains deterministic seasonal trigonometric terms. A parametric test of the null hypothesis that the component at a particular frequency is nonstationary against the alternative that it is stationary , that is $H_0 : \phi_j = 1$ against $H_1 : \phi_j < 1$ can be constructed from the null hypothesis innovations as

$$\omega_j = 2T^{-2} \sum_{i=1}^{T} \left[\left(\sum_{t=1}^{i} \widetilde{\nu}_t \cos\lambda_j t \right)^2 + \left(\sum_{t=1}^{i} \widetilde{\nu}_t \sin\lambda_j t \right)^2 \right] < c, \qquad j = 1, ..., [(s-1)/2]. \tag{3.13}$$

Under the null hypothesis the asymptotic distribution is $CvM_0(2)$ since if the nonstationary seasonal operator, $1 - 2cos\lambda_j L + L^2$, were to be applied it would remove the corresponding deterministic seasonal. For $j = s/2$

$$\omega_{s/2} = T^{-2} \sum_{i=1}^{T} \left(\sum_{t=1}^{i} \widetilde{\nu}_t \cos\lambda_j t \right)^2$$

and this has a $CvM_0(1)$ asymptotic distribution under the null. The full seasonal test statistic is formed[10] by summing the $\omega'_j s$ and its asymptotic distribution under the null is $CvM_0(s-1)$. With seasonal slopes the asymptotic distributions are $CvM_1(.)$; compare Smith and Taylor (1998).

[10] I conjecture that, if seasonal slopes are included, this is the LM test if the $\phi'_j s$ are the same for all j.

Seasonality tests based on an autoregreesive model, will tend to perform poorly in situations where an unobserved components model is appropriate. The simulation evidence in Hylleberg (1995) illustrates this point by looking at the results of using the HEGY test for moving average models, which, as Harvey and Scott (1994) note, typically arise as the reduced form of unobserved components models.

A rejection of the null hypothesis in a seasonal unit root test may be an indication of a deterministic seasonal component rather than a stationary seasonal component of the form (3.11); see the evidence in Canova and Hansen (1995, p 244). The appropriate test of the null of deterministic seasonality against the alternative of strong stationary seasonality, that is (3.11) with ϕ_j close to one, is, perhaps surprisingly, the same as the test against non-stationary seasonality described in sub-section 2.3; this follows form results in Harvey and Streibel (1998). This should be borne in mind when interpreting the results of seasonal stationarity and unit root tests.

3.4. Slope unit root test

The stochastic trend of (2.15) may be modified so as to give what is sometimes called a damped trend, that is

$$\begin{aligned} \mu_t &= \mu_{t-1} + \beta_{t-1} + \eta_t, & \eta_t &\sim NID(0, \sigma_\eta^2), \\ \beta_t &= \phi\beta_{t-1} + \zeta_t, & \zeta_t &\sim NID(0, \sigma_\zeta^2), \end{aligned}$$

If it is this component which appears in (3.2), a test of $H_0 : \phi = 1$ against $H_1 : \phi < 1$ is a unit root test on the slope. In the special case of the smooth trend model when $\sigma_\eta^2 = 0$, the test statistic is simply

$$\xi = T^{-1} \sum_{i=3}^{T} \left[\sum_{t=3}^{i} \Delta^2 y_t \right]^2 / \sum_{t=3}^{T} (\Delta^2 y_t)^2$$

The asymptotic distribution of this statistic is CvM_0. If (3.2) is generalised so as to contain a deterministic $p - th$ order polynomial trend, the residuals from a regression of $\Delta^2 y_t$ on a polynomial of order $p-2$ are used to form the test statistic which is then asymptotically CvM_{p-1}.

The stochastic trend component will not generally have σ_η^2 set to zero and it will usually appear in a model of the form (3.8), possibly with other components such as stochastic cycles and seasonals. A parametric test statistic may then be constructed from the innovations from the model fitted under the null hypothesis. The tests staistic is actually (3.7), but renamed ξ because what is now being tested is the null hypothesis of a second unit root.

4. Multivariate tests

4.1. Testing against a multivariate random walk

If $\mathbf{y}_t$ is a vector containing N time series the Gaussian multivariate local level model is

$$\begin{aligned} \mathbf{y}_t &= \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \sim NID(\mathbf{0}, \boldsymbol{\Sigma}_\epsilon), \\ \boldsymbol{\mu}_t &= \boldsymbol{\mu}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim NID(\mathbf{0}, \boldsymbol{\Sigma}_\eta), \quad t = 1, ..., T, \end{aligned} \tag{4.1}$$

where $\boldsymbol{\Sigma}_\varepsilon$ is an $N \times N$ positive definite (p.d.) matrix. Nyblom and Harvey (2000) show that an LBI test of the null hypothesis that $\boldsymbol{\Sigma}_\eta = \mathbf{0}$ can be constructed against the homogeneous alternative $\boldsymbol{\Sigma}_\eta = q\boldsymbol{\Sigma}_\varepsilon$. The test has the rejection region

$$\eta(N) = \text{ tr } \left[\mathbf{S}^{-1}\mathbf{C}\right] > c, \tag{4.2}$$

where

$$\mathbf{C} = T^{-2} \sum_{i=1}^{T} \left[\sum_{t=1}^{i} \mathbf{e}_t\right] \left[\sum_{t=1}^{i} \mathbf{e}_t\right]' \quad \text{and} \quad \mathbf{S} = T^{-1} \sum_{t=1}^{T} \mathbf{e}_t \mathbf{e}_t{}'. \tag{4.3}$$

where $\mathbf{e}_t = \mathbf{y}_t - \overline{\mathbf{y}}$. Under the null hypothesis, the limiting distribution of (4.2) is Cramér-von Mises with N degrees of freedom, $CvM(N)$. The distribution is $CvM_2(N)$ if the model contains a vector of time trends. Although the test maximizes the power against homogeneous alternatives, it is consistent against all nonnull $\boldsymbol{\Sigma}_\eta' s$ since $T^{-1}\eta(N)$ has a nondegenerate limiting distribution. This limiting distribution depends only on the rank of $\boldsymbol{\Sigma}_\eta$.

The $\eta(N)$ test can be generalized along the lines of the KPSS test quite straightforwardly as in Nyblom and Harvey (2000). Parametric adjustments can also be made by the procedure outlined for univariate models. This requires estimation under the alternative hypothesis, but is likely to lead to an increase in power. If there are no constraints across parameters, it may be more convenient to construct the test statistic, (4.2), using the innovations from fitted univariate models. Alternatively, the lagged dependent variable method of Leybourne and McCabe (1994) may be used. This is the approach taken by Kuo and Mikkola (2001) in their study of purchasing power parity. They conclude that dealing with serial correlation in this way leads to tests with higher power than those formed using the nonparametric correction.

In the above tests no restrictions are put on the matrices $\mathbf{S}$ and $\mathbf{C}$. If N is large this may be a problem- indeed $\mathbf{S}$ cannot be inverted for $N > T-$ and it may be necessary to assume some structure on the covariance matrices to reduce the number of parameters to be estimated. One possibility is to impose some spatial pattern. Panel methods may appear to offer a way out of this problem since if the units are mutually independent, the individual η statistics may be summed to give an overall test statistic which, by the central limit theorem is asymptotically normal under the null hypothesis. Alternatively the numerators and denominators may be summed separately as in Bhargava (1986, p 378-9). However, such statistics will be different from the above multivariate statistic unless $\mathbf{S}$ is diagonal and will be invalid with correlated units; see, for example, O'Connell (1998).

4.2. Testing for common trends

If the rank of $\boldsymbol{\Sigma}_\eta$ is K, the model has K common trends. Suppose we wish to test the null hypothesis that $rank(\boldsymbol{\Sigma}_\eta) = K$ against the alternative that $rank(\boldsymbol{\Sigma}_\eta) > K$ for $K = 1, .., N-1$. Let $\lambda_1 \geq \cdots \geq \lambda_N$ be the ordered eigenvalues of $\mathbf{S}^{-1}\mathbf{C}$. The $\eta(N)$ test statistic is the sum of these eigenvalues, but when the rank of $\boldsymbol{\Sigma}_\eta$ is $K^\dagger$, the limiting distribution of $T^{-1}\eta(N)$ is the limiting distribution of T^{-1} times the sum of the $K^\dagger$ largest eigenvalues. This suggests basing a test of the hypothesis that $rank(\boldsymbol{\Sigma}_\eta) = K$ on the sum of the $N - K$ smallest eigenvalues, that is

$$\eta(K, N) = \lambda_{K+1} + \cdots + \lambda_N, \quad K = 1, ..., N-1. \tag{4.4}$$

If $K^\dagger > K$ the relatively large values taken by the first $K^\dagger - K$ of these eigenvalues will tend to lead to the null hypothesis being rejected. This is the *common trends* test. Of course if we allow K to be zero, then $\eta(0, N) = \eta(N)$.

The distribution of the common trends test statistic under the null hypothesis is not of the Cramér-von Mises form but it does depend on functions of Brownian motion. The series expansion for $K = 1, ..., N-1$ is

$$\eta(K, N) \xrightarrow{d} \sum_{j=1}^{\infty} (\pi j)^{-2} \mathbf{u}_j' \mathbf{u}_j - \operatorname{tr} \left(\sum_{j=1}^{\infty} (\pi j)^{-3} \mathbf{u}_j \boldsymbol{v}_j' \right) \left(\sum_{j=1}^{\infty} (\pi j)^{-4} \boldsymbol{v}_j \boldsymbol{v}_j' \right)^{-1} \left(\sum_{j=1}^{\infty} (\pi j)^{-3} \boldsymbol{v}_j \mathbf{u}_j' \right) \tag{4.5}$$

where $\boldsymbol{v}_j$ and $\mathbf{u}_j$ are, respectively, $K \times 1$ and $(N-K) \times 1$ vectors which are mutually independent NID$(\mathbf{0}, \mathbf{I})$. The significance points for $\eta(K, N)$ depend on both K and N and are tabulated in Nyblom and Harvey (2000). A different set of critical values are used if the model has been extended to include time trends. Parametric and nonparametric adjustment for serial correlation may be made in the common trends test in much the same way as was suggested for the $\eta(N)$ test.

4.3. Multivariate unit root tests

Taylor and Sarno (1998, p288) generalise the ADF test by using the SURE technique to fit N equations of the form (3.1) with lagged first differences added to the right hand side. A multivariate Wald statistic, denoted MADF, is then formed to test the null hypothesis that all equations contain a unit root. They are not able to derive an asymptotic distribution for this test statistic. Abuaf and Jorion (1990) had earlier proposed a simplified test based on the assumption that the autoregressive parameter is the same in all equations, but even in this case no asymptotic distribution has been derived.

The UC model in (3.2) generalizes to

$$\mathbf{y}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} t + \boldsymbol{\mu}_t, \qquad \boldsymbol{\mu}_t = \boldsymbol{\phi}\boldsymbol{\mu}_{t-1} + \boldsymbol{\eta}_t, \quad t = 1, ..., T, \tag{4.6}$$

with $Var(\boldsymbol{\eta}_t) = \boldsymbol{\Sigma}_\eta$. As in the univariate case, residuals are formed by estimating the level and slope coefficients under the null hypothesis. Generalizing the test statistic (3.4) based on detrended observations yields

$$\zeta(N) = tr\left\{\frac{1}{T}\left[\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t\Delta\widetilde{\boldsymbol{\mu}}_t'\right]^{-1}\sum_{t=1}^{T}\widetilde{\boldsymbol{\mu}}_t\widetilde{\boldsymbol{\mu}}_t'\right\} \tag{4.7}$$

where $\widetilde{\boldsymbol{\mu}}_t = \mathbf{y}_t - \widetilde{\boldsymbol{\alpha}}_0 - \widetilde{\boldsymbol{\beta}}t$ for $t = 1, .., T$ and $\widetilde{\boldsymbol{\mu}}_0 = \mathbf{0}$ with $\widetilde{\boldsymbol{\beta}} = (\mathbf{y}_T - \mathbf{y}_1)/(T-1)$ and $\widetilde{\boldsymbol{\alpha}}_0 = \mathbf{y}_1 - \widetilde{\boldsymbol{\beta}}$. Writing $\zeta(N)$ in a form analogous to (4.2) makes it apparent that its asymptotic distribution under the null hypothesis is $CvM_1(N)$ with the lower tail defining the critical region. If there is no time trend the critical values are taken from the $CvM_0(N)$ distribution. The $\zeta(N)$ test is consistent but only against alternatives in which all the series are stationary. Like $\eta(N)$, the $\zeta(N)$ statistic is invariant to linear transformations of the data.

Now suppose that, as in Abuaf and Jorion (1990), $\boldsymbol{\phi} = \phi\mathbf{I}_N$ where ϕ is a scalar. The GLS estimator of $\phi - 1$ constructed from the observations detrended by setting ϕ equal to one is

$$\widetilde{\phi} - 1 = \sum_{t=2}^{T}\widetilde{\boldsymbol{\mu}}_{t-1}'\boldsymbol{\Sigma}_\eta^{-1}\Delta\widetilde{\boldsymbol{\mu}}_t / \sum_{t=2}^{T}\widetilde{\boldsymbol{\mu}}_{t-1}'\boldsymbol{\Sigma}_\eta^{-1}\widetilde{\boldsymbol{\mu}}_{t-1} \tag{4.8}$$

Provided the slope is included, a little algebraic manipulation, given in the appendix, shows that the numerator is constant and as a result $\zeta(N)$ is equal to $-N/\{2T(\widetilde{\phi} - 1)\}$. The LM test of the null hypothesis that $\phi = 1$ is based on the statistic

$$LM = (\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t'\boldsymbol{\Sigma}_\eta^{-1}\widetilde{\boldsymbol{\mu}}_{t-1})^2 / \sum_{t=1}^{T}\widetilde{\boldsymbol{\mu}}_{t-1}'\boldsymbol{\Sigma}_\eta^{-1}\widetilde{\boldsymbol{\mu}}_{t-1}$$

and this is also a monotonic transformation of $\zeta(N)$ being equal to $N^2/4\,\zeta(N)$. As in the univariate case, a one-sided test based on the lower tail of the distribution of $\zeta(N)$ means that the alternative is $\phi < 1$.

If the model is generalised to include more components a parametric test statistic can be constructed from the vector of standardized innovations. Corresponding to (3.7) this statistic is

$$\zeta(N) = tr\left\{T^{-2}\sum_{i=1}^{T}\left[\sum_{t=1}^{i}\widetilde{\boldsymbol{\nu}}_t\right]\left[\sum_{t=1}^{i}\widetilde{\boldsymbol{\nu}}'_t\right]\right\} = T^{-2}\sum_{i=1}^{T}\left[\sum_{t=1}^{i}\widetilde{\boldsymbol{\nu}}'_t\right]\left[\sum_{t=1}^{i}\widetilde{\boldsymbol{\nu}}_t\right]. \qquad (4.9)$$

If the innovations from fitted univariate models are used, the test statistic is of the form (4.2) so as to allow for cross-correlation.

Application to Stochastic Volatility- The multivariate stationarity and unit root test statistics for all four daily exchange rate series considered at the end of section 3 are $\eta(4) = 8.325$ and $\zeta(4) = 0.790$. Thus the stationarity test rejects the null hypothesis that there are no random walk components in the series, while the unit root test just rejects the null that all four series have unit roots at the 10% level of significance.[11] This is not inconsistent with the conclusions in Harvey *et al* (1994) and Nyblom and Harvey (2000) that the series have just two common trends.

4.4. Seasonal unit root tests

The seasonality tests can be generalised to multivariate series. For example, the multivariate test against nonstationary seasonality in N series will have a $CvM_1(Ns - N)$ distribution under the null hypothesis, while the seasonal unit test will be based on $CvM_0(Ns - N)$. A test of the null hypothesis that there is a certain number of common seasonal factors, corresponding to seasonal co-integration being present, can be obtained by generalizing the common trends test. However, this does require the construction of tables of critical values.

5. Tests when breaks are present

Suppose there is a structural break in the trend at a known time $\tau + 1$, and let $\lambda = \tau/T$ denote the fraction of the sample before the break occurs. Consider the following models:

[11] The 5% and 10% lower tail critical values for $CvM_0(4)$ are 0.641 and 0.796 respectively.

$$
\begin{array}{rrcl}
[1] & y_t & = & \mu_t + \delta w_t + \varepsilon_t \\
[2] & y_t & = & \mu_t + \beta t + \delta w_t + \delta_\beta (w_t t) + \varepsilon_t \\
[2a] & y_t & = & \mu_t + \beta t + \delta w_t + \varepsilon_t \\
[2b] & y_t & = & \mu_t + \beta t + \delta_\beta z_t + \varepsilon_t,
\end{array}
\tag{5.1}
$$

where μ_t is a random walk, ε_t is white noise, δ and δ_β are parameters and

$$
w_t = \begin{cases} 0 & \text{for } t \leq \tau \\ 1 & \text{for } t > \tau \end{cases} \quad \textit{and} \quad z_t = \begin{cases} 0 & \text{for } t \leq \tau \\ t - \tau & \text{for } t > \tau \end{cases}
$$

There is no slope in model [1] and so the only break is in the level. The other models all contain a time trend. In model [2] there is a structural change in both the level and the slope. Model [2*a*] contains a break in the level only while [2*b*] corresponds to a piecewise linear trend.

5.1. Stationarity tests

Under Gaussianity the LBI (and one-sided LM) test statistics for $H_0 : \sigma_\eta^2 = 0$ against $H_1 : \sigma_\eta^2 > 0$ in the above models are of the form (2.2), but have asymptotic distributions under the null hypothesis which depend on λ. Bearing in mind the additivity property of the Cramér-von Mises distribution noted in sub-section 2.5, Busetti and Harvey (2001) propose the following simplified test statistics for models [1] and [2]:

$$
\eta_i^* = \frac{\sum_{t=1}^{\tau} (\sum_{s=1}^{t} e_s)^2}{\tau^2 s^2} + \frac{\sum_{t=\tau+1}^{T} (\sum_{s=\tau+1}^{t} e_s)^2}{(T-\tau)^2 s^2}, \qquad i = 1, 2. \tag{5.2}
$$

The LBI statistics differ only insofar as the two parts of (5.2) receive weights of λ^2 and $(1-\lambda)^2$ respectively. The simplified statistics still depend on the location of the breakpoint, but their asymptotic distributions do not since

$$
\eta_i^* \Rightarrow \begin{cases} CvM_1(2) & \text{for } i = 1 \\ CvM_2(2) & \text{for } i = 2. \end{cases} \tag{5.3}
$$

Not having to consult a table giving the distribution of the test statistic for all the possible values of λ is a big advantage. Furthermore the tests immediately generalize to cases where there are several structural breaks. If there are

k breaks the distribution of the simplified statistic converges to a (second-level) generalized Cramér-von Mises distribution with $k+1$ degrees of freedom, that is $CvM_i(k+1)$, $i=1,2$. The Monte Carlo evidence presented in Busetti and Harvey (2001) indicates that the LBI test is clearly superior only in the region close to the null hypothesis and for break points near the beginning or end of the sample.

5.2. Unit root tests

The effects of breaks on LM type unit root tests can be analysed by taking first differences in (5.1). For level breaks, [1] and [2a], differencing creates a single outlier at time $\tau+1$. This may be removed by a 'pulse' dummy variable which takes the value one at $\tau+1$ and is zero otherwise. If the test statistics are constructed as in (3.4), their asymptotic distributions are unaffected - in terms of (3.5) all that happens is that $\Delta\widetilde{\mu}_{\tau+1}$ is zero. Thus

$$\zeta_i \Rightarrow \begin{cases} CvM_0(1) & \text{for } i=1 \\ CvM_1(1) & \text{for } i=2a. \end{cases} \tag{5.4}$$

The breaks in trend, on the other hand, do affect the distributions of the test statistics. Taking first differences of a piecewise linear trend, [2b], results in a level dummy variable being fitted from $\tau+1$ onwards. In model [2] a pulse at $\tau+1$ is also needed. However, in both cases the additivity property of the Cramér-von Mises distribution can be exploited so that statistics constructed in a similar way to those in (5.2) have $CvM_1(2)$ asymptotic distributions under the null, that is

$$\zeta_i^* = \frac{\sum_{t=1}^{\tau} \widetilde{\mu}_t^2}{\tau^2 s^2} + \frac{\sum_{t=\tau+1}^{T} \widetilde{\mu}_t^2}{(T-\tau)^2 s^2}, \qquad i=2,2b, \tag{5.5}$$

with

$$\zeta_i^* \Rightarrow \; CvM_1(2) \qquad \text{for } i=2,2b. \tag{5.6}$$

If the models are more general and parametric test statistics are constructed from innovations, estimation is carried out with the dummy variables in their original undifferenced form. The inclusion of the random walk component has the same effect as differencing.

5.3. Multivariate series and seasonality

Busetti and Harvey (2000) extend the Canova-Hansen test to allow for dummy variables modelling breaks in the seasonal pattern. A simplified test, constructed on the same basis as (5.2), has a $CvM_1(2s-2)$ asymptotic distribution when there is one such break. The asymptotic distributions of seasonal unit root tests, on the other hand, are not affected by the inclusion of seasonal break dummies since these become pulse variables under the null hypothesis.

Busetti (2001) extends the multivariate tests of sub-sections 4.1 and 4.2 to deal with situations where there are breaks in some or all of a set of N time series. He shows that a simplified version of the test against a multivariate random walk can be constructed by allowing for a break in all the series at the same point in time. This statistic, denoted $\eta_i^*(N)$, generalises (5.2) and has the $CvM(2N)$ asymptotic distribution. The modification of multivariate unit root tests follows along similar lines to yield a generalisation of (5.5).

6. Conclusions

Unit root tests can be set up using the LM principle so as to give statistics which, under the null hypothesis, have Cramér-von Mises distributions in large samples. Stationarity test statistics have asymptotic distributions belonging to the same family. This provides a remarkable unification and simplification of test procedures for nonstationary time series. The distributions are easily tabulated and have nice properties, such as additivity. For the simpler models exact distributions of the test statistics can be obtained, but once nuisance parameters have to be estimated, the case for just using the asymptotic distributions becomes stronger. In any case it seems that the asymptotic critical values provide a good approximation even for relatively small sample sizes. The additivity property of the Cramér-von Mises distribution means that it is easy to set up tests with an allowance made for any intervention variables used to model structural breaks.

The tests are obtained by working within an unobserved components framework. There is a strong case for estimating the nuisance parameters in such models and constructing parametric tests. Autoregressive approximations and nonparametric estimates of the long-run variance can often lead to tests with unreliable type I errors and/or low power. Modifications could be made to the tests along the lines suggested by Elliott, Rothenberg and Stock (1996) or Hwang and Schmidt (1996), but this would be at the cost of losing the simplicity and generality of the test statistics and their asymptotic distributions.

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A. Appendix

Let T times the coefficient obtained by regressing $\Delta\widetilde{\mu}_t$ on $\widetilde{\mu}_{t-1}$ without a constant term be denoted by $\overline{\rho}$. To show the relationship with ζ, write the denominator of (3.5) as

$$\begin{aligned}\sum_{t=1}^{T}(\widetilde{\mu}_t-\widetilde{\mu}_{t-1})^2 &= \sum_{t=1}^{T}\widetilde{\mu}_t^2+\sum_{t=1}^{T}\widetilde{\mu}_{t-1}{}^2-2\sum_{t=1}^{T}\widetilde{\mu}_t\widetilde{\mu}_{t-1}\\ &= 2\left[\sum_{t=1}^{T}\widetilde{\mu}_{t-1}^2-\sum_{t=1}^{T}\widetilde{\mu}_t\widetilde{\mu}_{t-1}\right]+\widetilde{\mu}_T^2=-2\sum_{t=2}^{T}\Delta\widetilde{\mu}_t\widetilde{\mu}_{t-1}+\widetilde{\mu}_T^2\end{aligned}$$

This uses the fact that $\widetilde{\mu}_0$ is always zero. Note that if the summation starts at $t=2$ then

$$\sum_{t=2}^{T}(\widetilde{\mu}_t-\widetilde{\mu}_{t-1})^2=-2\sum_{t=2}^{T}\Delta\widetilde{\mu}_t\widetilde{\mu}_{t-1}+\widetilde{\mu}_T^2-\widetilde{\mu}_1^2$$

With a constant $\widetilde{\mu}_1=0$ and with a time trend as well $\widetilde{\mu}_T=0$ so that $\zeta=-1/2\overline{\rho}$.

In the multivariate model, $\overline{\rho}$ is $T(\widetilde{\phi}-1)$ in (4.8). Applying the same argument gives

$$\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t'\boldsymbol{\Sigma}_\eta^{-1}\Delta\widetilde{\boldsymbol{\mu}}_t=-2\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t'\boldsymbol{\Sigma}_\eta^{-1}\widetilde{\boldsymbol{\mu}}_{t-1}+\widetilde{\boldsymbol{\mu}}_T'\boldsymbol{\Sigma}_\eta^{-1}\widetilde{\boldsymbol{\mu}}_T$$

If $\boldsymbol{\Sigma}_\eta$ is estimated by $T^{-1}\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t\Delta\widetilde{\boldsymbol{\mu}}_t'$, the left hand side of the above expression reduces to TN because $\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t'\boldsymbol{\Sigma}_\eta^{-1}\Delta\widetilde{\boldsymbol{\mu}}_t=tr\left[\boldsymbol{\Sigma}_\eta^{-1}\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t\Delta\widetilde{\boldsymbol{\mu}}_t'\right]$ and so, provided the slope is estimated, it follows that $\zeta(N)=-N/2\overline{\rho}$. As regards the LM test, evaluating the first derivative of the log-likelihood function at $\phi=1$ yields

$$\frac{\partial\log L}{\partial\phi}=\sum_{t=1}^{T}(\widetilde{\boldsymbol{\mu}}_t'-\phi\widetilde{\boldsymbol{\mu}}_{t-1})'\boldsymbol{\Sigma}_\eta^{-1}\widetilde{\boldsymbol{\mu}}_{t-1}=\sum_{t=1}^{T}\Delta\widetilde{\boldsymbol{\mu}}_t'\boldsymbol{\Sigma}_\eta^{-1}\widetilde{\boldsymbol{\mu}}_{t-1}=-1/2NT.$$

On evaluating the second derivative we find that $\zeta(N)=N^2/4LM$.

REFERENCES

Abuaf, N. and P. Jorion (1990). Purchasing power parity in the long run. *Journal of Finance* 45, 157-74.

Bhargava, A.(1986), On the theory of testing for unit roots in observed time series, *Review of Economic Studies,* 53, 36-84.

Busetti, F.and A.C. Harvey (2001), Testing for the presence of a random walk in series with structural breaks. *Journal of Time Series Analysis,* **22,** 127-50.

Busetti, F.and A.C. Harvey (2000), Testing against stochastic seasonality. Mimeo.

Busetti, F (2001), Testing for (common) nonstationary components in multivariate time series with breaking trends. *Journal of Forecasting (*to appear).

Canova, F. and B.E. Hansen (1995), Are seasonal patterns constant over time? A test for seasonal stability, *Journal of Business and Economic Statistics,* **13,** 237-52.

O'Connell, P. (1998), The overvaluation of purchasing power parity, *Journal of International Economics,* 44, 1-19.

Elliott, G., Rothenberg, T.J. and J.H.Stock (1996). Efficient tests for an autoregressive unit root. *Econometrica,* 64, 813-36.

Fuller, W. A. (1996). *Introduction to Statistical Time Series*, 2nd edition. John Wiley and sons, Inc., New York.

Harvey A.C. (1989), *Forecasting, structural time series models and the Kalman filter*, Cambridge University Press, Cambridge.

Harvey A.C., E. Ruiz and N. Shephard (1994), Multivariate stochastic variance models, *Review of Economic Studies*, **61**, 247-64.

Harvey, A. C., and A. Scott (1994). Seasonality in dynamic regression models. *Economic Journal* 104, 1324-45.

Harvey, A.C. and M. Streibel (1997), Testing for nonstationary unobserved components, mimeo.

Harvey, A. C., and M. Streibel (1998). Tests for deterministic versus indeterministic cycles. *Journal of Time Series Analysis* 19: 505-29.

Hwang, J. and P. Schmidt (1996), Alternative methods of detrending and the power of unit root tests. *Journal of Econometrics* 71, 227-8

Hylleberg, S. (1995), Tests for seasonal unit roots: general to specific or specific to general? *Journal of Econometrics*, 69, 5-25.

Hylleberg, S., Engle, R., Granger, C.W.J and B.S. Yoo (1990), Seasonal integration and co-integration, *Journal of Econometrics* 44, 215-38.

Koopman S.J., Harvey, A.C., Doornik, J.A and Shephard, N. (2000). *STAMP 6.0 Structural Time Series Analyser, Modeller and Predictor*, London: Timberlake Consultants Ltd.

Kuo, B-S. and A. Mikkola (2001), How sure are we about purchasing power parity? Panel evidence with the null of stationary exchange rates. *Journal of Money Credit and Banking*

Kwiatkowski, D., Phillips, P.C.B, Schmidt, P. and Y.Shin (1992), Testing the null hypothesis of stationarity against the alternative of a unit root: How sure are we that economic time series have a unit root ? *Journal of Econometrics* 44, 159-78.

Kuo, B-S. (1999), Asymptotics of ML estimator for regression models with a stochastic trend component. *Econometric Theory,* **15,** 24-49.

Leybourne, S.J. and B.P.M. McCabe (1994), A consistent test for a unit root, *Journal of Business and Economic Statistics,* **12**, 157-66.

MacNeill, I.B. (1978), Properties of sequences of partial sums of polynomial regression residuals with applications to tests for change of regression at unknown times, *The Annals of Statistics,* **6**, 422–433.

Maddala, G.S. and I-M. Kim (1998). *Unit roots, co-integration, and structural change.* Cambridge: Cambridge University Press.

Nabeya, S. and K. Tanaka (1990), Limiting power of unit-root tests in time-series regression, *Journal of Econometrics,* **46,** 247-71.

Nyblom, J. and T. Mäkeläinen (1983), Comparison of tests for the presence of random walk coefficients in a simple linear model, *Journal of the American Statistical Association, **78**, 856-64.*

Nyblom, J., and A.C.Harvey (2000). Tests of common stochastic trends, *Econometric Theory,* 16, 176-99.

Nyblom, J., and A.C.Harvey (2001). Testing against smooth stochastic trends, *Journal of Applied Econometrics (to appear).*

O'Connell, P. (1998), The overvaluation of purchasing power parity, *Journal of International Economics,* 44, 1-19.

Oya, K. and H.Y.Toda (1998), Dickey-Fuller, Lagrange multiplier and combined tests for a unit root in autoregressive time series. *Journal of Time Series Analysis* 19, 325-47.

Pantula, S. (1991). Asymptotic distributions of unit root tests when the process is nearly stationary. *Journal of Business and Economic Statistics*, 9, 63-71.

Perron, P. and S.Mallet (1996), A look at the quality of the approximation of the functional central limit theorem, *Economics Letters,* 68, 225-34.

Rodrigues, P.M.M. (2000), On LM type tests for seasonal unit roots, mimeo, University of Algarve.

Schmidt, P. and P.C.B. Phillips (1992): LM Tests for a Unit Root in the Presence of Deterministic Trends. *Oxford Bulletin of Economics and Statistics* 54, 257-287.

Schmidt, P. and J. Lee (1991): A modification of the Schmidt-Phillips unit root test, *Economics Letters,* **36,** 285-89.

Smith, R.J. and A.M.R. Taylor (1998), Additional critical values and asymptotic representations for seasonal unit root tests, *Journal of Econometrics* 85, 269-288.

Tanaka, K.(1996). *Time Series Analysis.* New York: John Wiley and Sons.

Taylor, M.P. and L. Sarno (1998), The behaviour of real exchange rates during the post-Bretton Woods period. *Journal of International Economics,* **46,** 281-312.