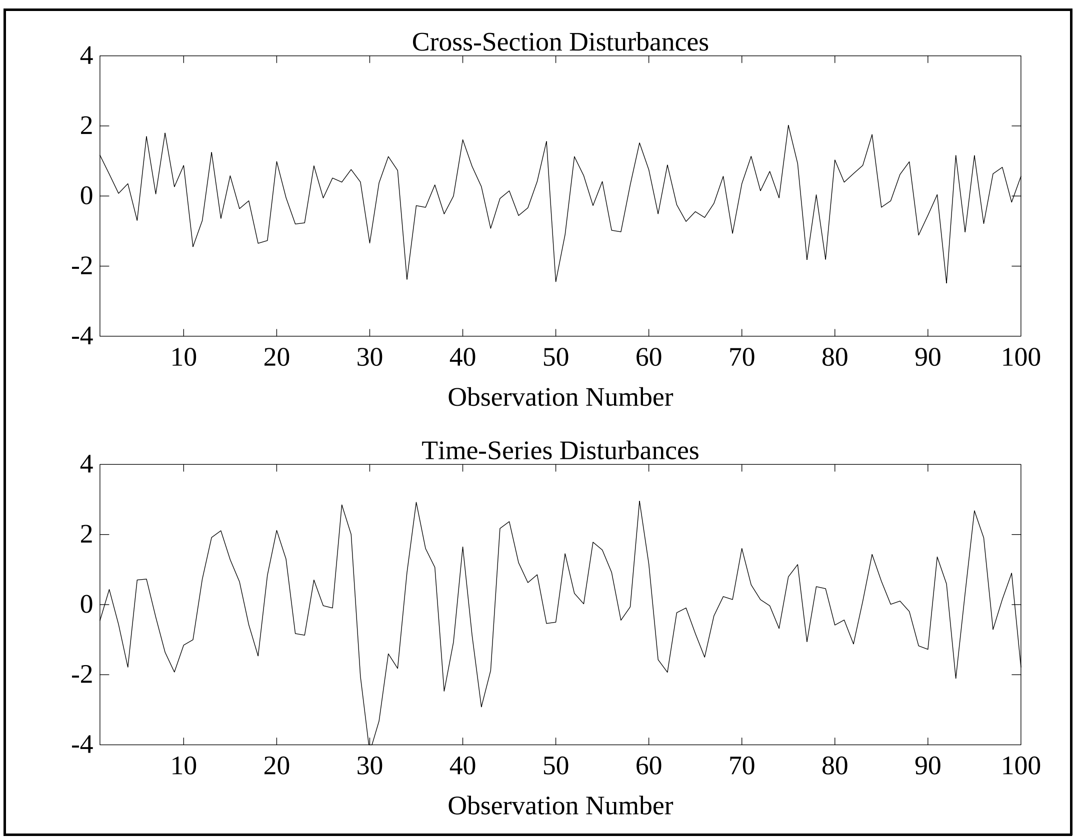
LECTURE / DISCUSSION

Time Series Models, Part 1

Introduction

Consider the application of OLS regression methods to a time series on the revenues of a utility. The explanatory variables would naturally include such macroeconomic time series as a price index, interest rate, an unemployment rate, and a measure of aggregate demand. A measure of productivity would be another natural explanatory variable, as would dummy variables for time periods containing different regulatory rules governing the utility. After including these explanatory variables, what concerns would you have about the properties of the disturbance term? Most econometricians would answer that the disturbances are unlikely to be independently distributed over the observations. Generally speaking, the present, which is influenced by the past and the disturbance term (that contains all of the omitted explanatory variables), is likely to exhibit this phenomenon.

Look at the differences between graphs of fitted residuals from a cross-section regression and a time-series regression, normalized so that both have the same sample variance. You can see that the time-series of residuals, for which the observations have a natural order, have a smoother appearance. This ‘‘smoothness” appears because positive residuals tend to be preceded or followed by positive residuals more frequently in the time series than in the cross-section. Negative residuals behave likewise. In this way, it appears that knowing a disturbance at any point in time would help in predicting the next disturbance. If so, then the disturbances are correlated.



Example
Serial Correlation

[Graphic not available for posting.]

Residuals from Log (Revenue) Regression

[Graphic not available for posting.]

In this lecture, we will

- introduce the simplest model of correlation among the disturbances of a regression equation,
- describe the implications for estimation and statistical inference, and
- explain a basic method for detecting evidence of such correlation.

Autoregressive Disturbances: The AR(1) Model

In the spirit of the linear regression model, correlation among disturbances can be modelled by the simple ***autoregressive*** form

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad |\rho| < 1$$

where the ε_t are i.i.d. normal random variables with mean zero and variance σ_ε^2. This kind of disturbance is called ***first-order*** autoregressive, because only the previous value of u_t appears as an explanatory variable of u_t. In a slight abuse of language, u is ‘‘regressed’’ on itself.

The properties of the sequence of u_t are:

1. Mean zero: $E(u_t) = 0$

2. Constant variance: $Var(u_t) = \sigma_\varepsilon^2 \ / \ (1 - \rho^2)$

3. Autocorrelated: $Cov(u_t, u_{t-s}) = \rho^{|s|} Var(u_t)$

4. Normally distributed

The probabilistic dependence among the disturbances is captured by the functional dependence of u_t on u_{t-1}. Through u_{t-1}, u_t depends on all of the previous values of u: Because u_{t-1} depends in turn on u_{t-2}, it follows that u_t does also. This recursive relationship continues on into the indefinite past. But the dependence diminishes as the separation in time grows (because $|\rho| < 1$):

$$Cov(u_t, u_{t-s}) = \rho^{|s|} Var(u_t) \rightarrow 0 \text{ as } s \rightarrow \pm \infty$$

If ρ were not less than one in absolute value, then this model would be inappropriate for most time-series problems. We will return to this issue in our next lecture on time-series models. In most applications, we find $\rho > 0$, reflecting inertia in the disturbances.

Effects on the Properties of OLS Estimates

If $\rho = 0$, then the original, classical model holds: The correlation among the disturbances vanishes and all of the properties of OLS that we described earlier hold. If $\rho \neq 0$, some of these properties fail. Because the expectation of the dependent variable remains $x_t'\beta$, OLS estimates of regression coefficients remain unbiased. However, because the covariances of the dependent variable are altered, the properties related to variances fail:

1. OLS coefficient estimates are not ***minimum variance*** linear unbiased estimators.

2. OLS standard errors are biased estimators of the standard deviations of the OLS coefficient estimators.

Somewhat surprisingly, the OLS estimator of the variance of the disturbances is still a useful estimator.

Testing for AR(1) Disturbances

Given the potential failures of OLS procedures when disturbances are correlated, statistical tests for correlation are commonly applied to time-series regressions. A useful approach to constructing a test for correlation is to consider what one would do if the u_t were actually observable. The same procedure, or a slight modification of it, will often work when the OLS fitted residuals $\hat{u}_t$ are substituted for the u_t.

If the u_t were observable, we could test $H_0 : \rho = 0$ with a simple *t* test after regressing u_t on u_{t-1}.

The same test procedure, made feasible by regressing $\hat{u}_t$ on $\hat{u}_{t-1}$ also works. It is essentially the same testing procedure as the Durbin-Watson test. The Durbin-Watson test statistic is printed out automatically by most software with every regression run.

Durbin-Watson Statistic

Tests for serial correlation.

0	d_L	d_U	2
Serial corr.	Inconclusive	No serial corr.	

$$DW = \frac{\sum (\hat{u}_t - \hat{u}_{t-1})^2}{\sum (\hat{u}_t)^2}$$

When errors are serially correlated, u_t is close to u_{t-1} and so numerator is small. When errors have no serial correlation, numerator has expectation

$$\sum (\hat{u}_t)^2 + \sum (\hat{u}_{t-1})^2 \quad ,$$

which is twice the denominator.

So, DW ranges from 0 (perfect serial correlation) to 2 (no serial correlation).

Allowing for negative serial correlation:

0	d_L	d_U	2	$4-d_U$	$4-d_L$	4
Pos. serial corr.	Inconclusive	No serial corr.		Inconclusive	Neg. Serial corr.	

Example

log (Rev) = -3.62 + 1.66 log (Income)
(.43)   (.06)

DW = 0.223
K = number of variables (in this case, K = 1)
N = 34
d_L = 1.39 , d_U = 1.51 at 5% significance

Regressing $\hat{u}_t$ on $\hat{u}_{t-1}$,

$\hat{\rho} = 0.896$
standard error = 0.0835
t-statistic = 10.730

☞ Conclude that there is positive correlation.

Note:

$$2(1-\hat{\rho}) = 0.208 \cong DW$$

Correcting for AR(1) Disturbances

A regression of u_t on u_{t-1} will yield an unbiased estimator of the unknown ρ. Fortunately, the feasible regression of $\hat{u}_t$ on $\hat{u}_{t-1}$ will also yield an estimator of ρ, and this estimator is the basis of methods for correcting standard errors and lowering the variance of our estimates of β. If there is strong evidence that the disturbances are correlated, then one will want to make these corrections.

To understand the method, consider another counterfactual situation: Suppose that ρ were known. How would you proceed? A fundamental insight is that one can transform the regression model with AR(1) disturbances into a regression model that satisfies the conditions of the original OLS theory.

To do this note that according to the AR(1) model,

$$\varepsilon_t = u_t - \rho u_{t-1}$$

is a normal, i.i.d. disturbance term. If ε_t were the disturbance term of a regression model, we could apply all of our OLS tools. And it can be:

$$\begin{aligned}\varepsilon_t &= u_t - \rho u_{t-1} \\ &= (y_t - x_t'\beta) - \rho(y_{t-1} - x_{t-1}'\beta)\end{aligned}$$

or

$$y_t - \rho y_{t-1} = (x_t - \rho x_{t-1})'\beta + \varepsilon_t$$

has the desired form. Instead of regressing y_t on x_t, we must regress the ***quasi first-difference*** $y_t - \rho y_{t-1}$ on the quasi-first-difference $x_t - \rho x_{t-1}$.

We make this procedure feasible by substituting for ρ an estimator based on fitted OLS residuals.

Example:

$$\log(\text{Revenue}) = \underset{(1.41)}{-0.26} + \underset{(.20)}{1.18}\log(\text{Income})$$

$$\hat{\rho} = 0.98 \quad (.02)$$

Note: AR1 procedure in TSP estimates β and ρ iteratively.

1. Estimate β with original variables.
2. Estimate ρ by regressing residuals against lagged residuals.
3. Create quasi-first-differences using the estimate of ρ from step 2.
4. Estimate β using the quasi-first-differences in step 3.
5. Create residuals using β from step 4.
6. Estimate ρ by regressing residuals from step 5 against their lagged values.
7. Create quasi-first-differences using the estimate of ρ from step 6.

And so on, until estimates of ρ and β do not change.

Residuals from Log (Revenue) AR1 Regression

(Corrected for First-Order Serial Correlation)

[Graphic not available for posting.]