Problem Set 6
Suggested Solutions

1. Walras' Law: $\sum_{i=1}^{N} p_i z_i \left(\underline{p} \right) = 0$

<u>Proof:</u>   Assume that each agent chooses optimally to consume a bundle that lies on his budget constrain. We have:

$$\sum_{i=1}^{N} p_i x_i^h \left(\underline{p} \right) = \sum_{i=1}^{N} p_i e_i^h \Leftrightarrow \sum_{i=1}^{N} p_i \left(x_i^h \left(\underline{p} \right) - e_i^h \right) = 0 \qquad \forall h \in \{1,...,H\}$$

Summing over all the agents:

$$\sum_{h=1}^{H} \sum_{i=1}^{N} p_i \left(x_i^h \left(\underline{p} \right) - e_i^h \right) = 0 \Leftrightarrow \sum_{i=1}^{N} \sum_{h=1}^{H} p_i \left(x_i^h \left(\underline{p} \right) - e_i^h \right) = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \sum_{h=1}^{H} p_i z_i^h \left(\underline{p} \right) = 0 \Leftrightarrow \sum_{i=1}^{N} p_i z_i \left(\underline{p} \right) = 0$$

It is clear from the above proof, that whether or not $\underline{p} = \left(p_i \right)_{i=1}^{N}$ and $\underline{x} = \left[x_i^h \right]_{\substack{i=1,...,N \\ h=1,...,H}}$ is a

Walrasian equilibrium is not used in the proof. The only requirement is that every agent's choice lies on the budget frontier (which can be met by assuming strict monotonicity of preferences in at least one commodity or local non-satiation). If this is the case, Walras' Law will hold even at a non-WE point (see Fig. 1)[1].
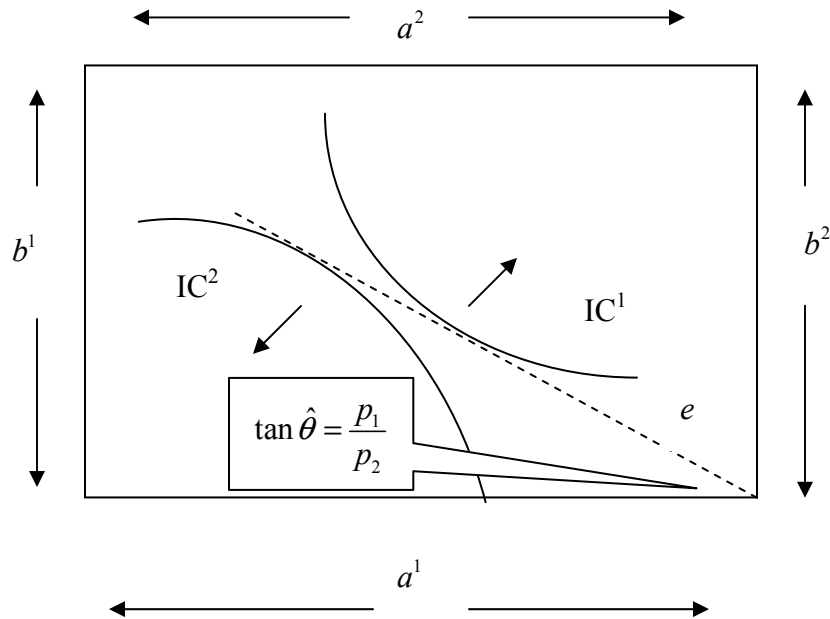
$$\tan \hat{\theta} = \frac{p_1}{p_2}$$

Fig. 1

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2.   (TRUE)   This is essentially a restatement of the second welfare theorem. The required incomes policy is one of lump sum transfers between agents (that are to take place before the beginning of trading) so that the new endowment vector is the original WEA. Following the transfer period, trade will be allowed but, of course, no trade will actually take place in equilibrium.

3.   (FALSE)   If the initial endowment is on the contract curve, then it is a pareto efficient allocation. This does not necessarily mean, however, that it can be supported by a strictly positive vector of prices as a WEA.
An example of such a situation is PS#5/Q1(A). In that problem, we showed that, only points on the western and northern borders of the Edgeworth box (excluding the SW and NE corners) could be on the contract curve because these were the only points where at least one agent was optimizing. Yet the unique WE was at the northwest corner of the box. Any other point of the western and northern borders, if it were given as the endowment point, would clearly be on the contract curve[2]. Yet it could not be a WEA because the two indifference curves are not tangent at that point[3].

You might be puzzled here by this answer seeming inconsistent with the one for the preceding question. The catch is in the fact that, in question 2, we are assuming that the requirements for the second welfare theorem to apply are met. Here we are _not_ told that we may do so and, hence, we may not appeal to the theorem. In particular with respect to the example in PS#5/Q1(A), the assumption of strict quasi-concavity of the utility functions (from the set of assumptions that JR have termed assumption 5.1 (pp.188)) is violated. The given utility functions $U^1 = a^1 + \log b^1$, $U^2 = \log a^2 + b^2$ are quasi-concave but not strictly so[4].

4.   (TRUE) Notice that only L and F respectively enter the utility functions of Jack and his wife. Therefore, given a price vector $\underline{p} = (p_1, p_2)$, Jack is facing the budget constraint $p_1 F + p_2 L \leq 70 p_1 + 35 p_2$ and he will maximize his welfare by trading his entire endowment of F for as much L as possible. Hence, he will consume at

$$\left(F^J, L^J \right) = \left(0, \frac{70 p_1 + 35 p_2}{p_2} \right).$$

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[2] Since one agent is optimizing and the other (being at his endowment point) is doing at least as well as at his endowment point.

[3] Note that, from the perspective of an outside observer, a point on the western border is a corner point for the Edgeworth box. However, from the perspective of the agent, it is not. Especially when that point becomes an endowment point, it falls right in the middle of his budget constraint. Hence, the only way for the agent to be optimizing at that point is for his indifference curve to be tangent to the budget line. But, given that the two agents must be facing the same price ratio, the two indifference curves having different slopes renders impossible them to be optimizing at the same time as the WEA requires.

[4] In general, this means that we are not guaranteed to find a WEA with strictly positive prices for all commodities. In the given example, we cannot find equilibrium prices at all since, in order to be optimizing at a western or northern border point, one agent needs the price ratio to be different than 1 whereas the other agent needs it to be 1.

Similarly, Jack's wife will be choosing: $\left(F^W, L^W\right) = \left(\dfrac{42p_1 + 70p_2}{p_1}, 0\right)$

Clearly, a Pareto Optimal allocation will be at *the* corner of the Edgeworth box where each agent gets allocated the entire endowment of the single good that enters his utility function. At any other point of the box, we can find an allocation that pareto-dominates the given point by giving all of F^J to the wife and all of L^W to Jack. Neither agent is worse off since we are taking away from him/her an amount of a commodity that does not enter their utility function anyway.

5.   Notice that, for the first group of agents, the utility function is linear in both arguments with $MU_{x_1} = MU_{x_2}$. Since, each commodity contributes equally to the level of utility, the agent is indifferent between consuming either commodity (i.e. all vectors $(x_1, x_2) \in R_+^2 : x_1 + x_2 = c$ give the same level of utility c). Hence, the demand of agents in the first group will be:

$$\left(x_1^I, x_2^I\right) = \begin{cases} \left(0, \dfrac{w^I}{p_2}\right) & \text{if } p_1 > p_2 \\[2ex] \left(\dfrac{w^I}{p_1}, 0\right) & \text{if } p_1 < p_2 \\[2ex] (x_1, x_2) : x_1 + x_2 = \dfrac{w^I}{p_1} & \text{if } p_1 = p_2 \end{cases} \qquad (1)$$

For the second group, the indifference curves are right angles stemming from the origin having their vertices on the curve $2x_1 = x_2$ (see Fig.5).
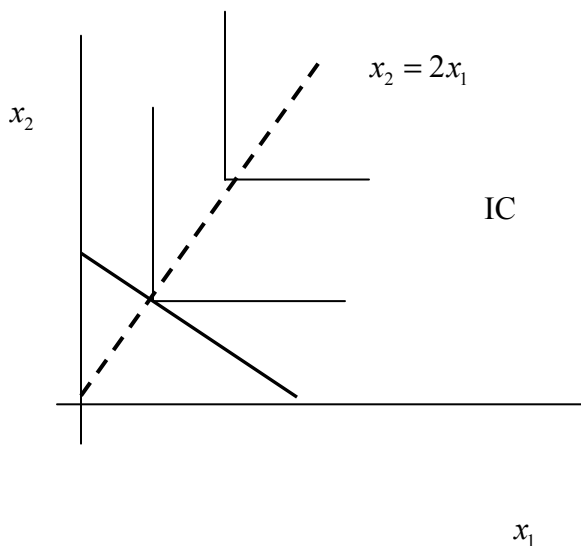

x_2

$x_2 = 2x_1$

IC

a_0^1

Fig. 5

x_1

x_1

Hence, the optimal bundles will be given by the intersection of this curve with the budget line that the representative agent from this group faces:

$$2x_1 = x_2 \text{ and } p_1x_1 + p_2x_2 = w$$

Hence,

$$\left(x_1^{II}, x_2^{II}\right) = \left(\frac{w^{II}}{p_1 + 2p_2}, \frac{2w^{II}}{p_1 + 2p_2}\right) \qquad (2)$$

Moreover, to get a WE, we need to make sure that the allocation is feasible:

$$x_i^{I} + x_i^{II} = e_i^{I} + e_i^{II} \qquad \text{for } i \in \{1,2\} \qquad (3)$$

Solving the system (1)-(3), for $w^I = w^{II} = p_1 + p_2$ we find:

If $p_1 > p_2$, equation (3) for $i = 1$ gives: $p_1 + 3p_2 = 0$. Hence, we cannot have a solution in this case.

If $p_1 < p_2$, equation (3) for $i = 2$ gives: $p_2 = 2p_2$ and, thus, we cannot have a solution in this case either.

If $p_1 = p_2$, equation (2) gives: $\left(x_1^{II}, x_2^{II}\right) = \left(\frac{2}{3}, \frac{4}{3}\right)$ while equation (1) gives $x_1^{I} + x_2^{I} = 2$.

Hence, $\left(x_1^{I}, x_2^{II}\right) = \left(\frac{4}{3}, \frac{2}{3}\right)$.

We conclude that the set of WE is:

$$\left\{\left((p_1, p_2), (x_1^{I}, x_2^{I}), (x_1^{II}, x_2^{II})\right)\right\} = \left\{\left((p,p), \left(\frac{4}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{4}{3}\right)\right): p > 0\right\}$$

6. Consider the firm's optimization problem

$$\max_{(y,x_1,x_2)} \pi = py - w_1x_1 - w_2x_2$$

$$s.t.$$

$$y \le f(x_1, x_2) = \sqrt{2x_1 + 4x_2}$$

$$x_1, x_2, y \ge 0$$

The Lagrangean:

$$L(x_1, x_2, y; \lambda, \mu_1, \mu_2, \mu_3) = py - w_1x_1 - w_2x_2 + \lambda\left[\sqrt{2x_1 + 4x_2} - y\right]$$
$$+ \mu_1(x_1 - 0) + \mu_2(x_2 - 0) + \mu_3(y - 0)$$

The first-order conditions:

Set (I)

$$\frac{\partial \pi (y, \underline{x})}{\partial y} = \lambda - \mu_3 \quad \Leftrightarrow \quad \lambda - \mu_3 = p \qquad\qquad (1.1)$$

$$\frac{\partial \pi (y, \underline{x})}{\partial x_1} = \frac{\lambda}{\sqrt{2x_1 + 4x_2}} + \mu_1 \quad \Leftrightarrow \quad \frac{\lambda}{\sqrt{2x_1 + 4x_2}} + \mu_1 = w_1 \qquad (1.2)$$

$$\frac{\partial \pi (y, \underline{x})}{\partial x_2} = \frac{2\lambda}{\sqrt{2x_1 + 4x_2}} + \mu_2 \quad \Leftrightarrow \quad \frac{2\lambda}{\sqrt{2x_1 + 4x_2}} + \mu_2 = w_2 \qquad (1.3)$$

Set (II)

$$y \leq \sqrt{2x_1 + 4x_2} \qquad\qquad (2.1)$$
$$\lambda \geq 0 \qquad\qquad (2.2)$$
$$\lambda \left[\sqrt{2x_1 + 4x_2} - y \right] = 0 \qquad\qquad (2.3)$$

$$x_1 \geq 0 \qquad \mu_1 \geq 0 \qquad \mu_1 x_1 = 0 \qquad (3.1)$$

$$x_2 \geq 0 \qquad \mu_2 \geq 0 \qquad \mu_2 x_2 = 0 \qquad (3.2)$$

$$y \geq 0 \qquad \mu_3 \geq 0 \qquad \mu_3 y = 0 \qquad (3.3)$$

Note, however, that firm would never choose to produce a zero level of output. Hence, it must be $y > 0$ and, from (3.3), $\mu_3 = 0$.
From (1.1), it is now obvious that $\lambda = p > 0$ and, consequently, the firm must be operating on the production possibility frontier.
In other words, $y = \sqrt{2x_1 + 4x_2}$

For an interior solution, we can now simplify the original problem greatly by ignoring the non-negativity constraints on y, x_1, x_2 and substituting $y = \sqrt{2x_1 + 4x_2}$ into the objective. The simplified problem becomes:
$$\max_{(x_1, x_2)} \pi = p\sqrt{2x_1 + 4x_2} - w_1 x_1 - w_2 x_2$$

The first order conditions, for an interior solution, are given:
$$\frac{\partial \pi (\underline{x})}{\partial x_1} = 0 \Leftrightarrow \frac{p}{\sqrt{2x_1 + 4x_2}} = w_1 \qquad (1.2.i)$$

5

$$\frac{\partial \pi(\underline{x})}{\partial x_2} = 0 \Leftrightarrow \frac{2p}{\sqrt{2x_1 + 4x_2}} = w_2 \qquad (1.3.\text{i})$$

We have: $\qquad \dfrac{(1.2.i)}{(1.3.i)} \to 2w_1 = w_2 \qquad$ and $\qquad \sqrt{2x_1 + 4x_2} = \dfrac{p}{w_1}$

The combinations of inputs are $\left\{ (x_1, x_2) : \sqrt{2x_1 + 4x_2} = \dfrac{w_1}{p} \Leftrightarrow x_1 = \dfrac{1}{2}\left(\dfrac{w_1}{p}\right)^2 - 2x_2 \right\}$

Let us now check the corner solutions:

Consider $x_1 = 0, \ x_2 > 0$

From (3.2): $\mu_2 = 0$

From (1.3): $\qquad x_2 = \left(\dfrac{p}{w_2}\right)^2$

From (1.2) $\mu_1 = w_1 - \dfrac{p}{2\sqrt{x_2}} = w_1 - \dfrac{w_2}{2} \qquad\qquad$ Hence, for $\mu_1 > 0$, we require $2w_1 > w_2$
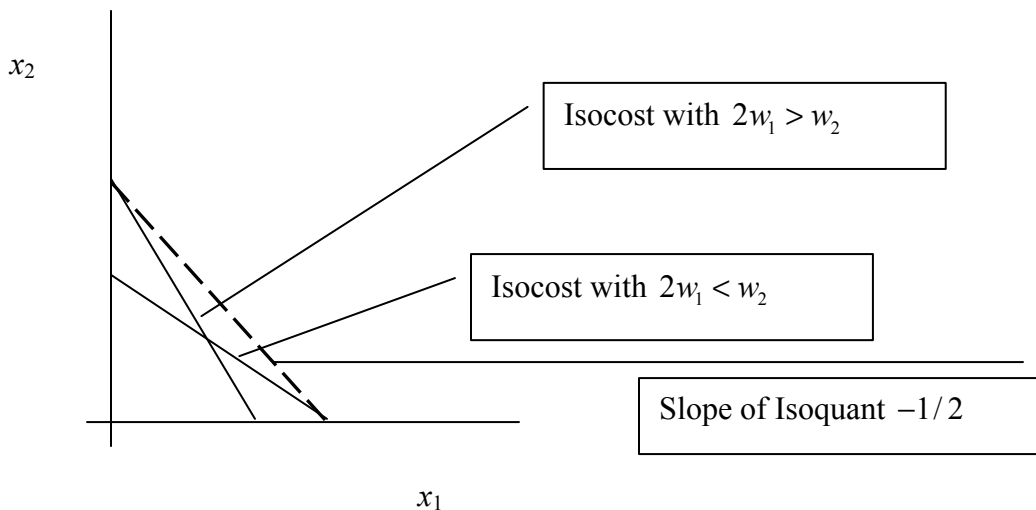
Consider $x_1 > 0, \ x_2 = 0$

From (3.1): $\mu_1 = 0$

From (1.2): $\qquad x_1 = \dfrac{1}{2}\left(\dfrac{p}{w_1}\right)^2$

From (1.3) $\mu_2 = w_2 - \dfrac{2p}{\sqrt{2x_1}} = w_2 - 2w_1$. Hence, for $\mu_2 > 0$, we require $2w_1 < w_2$

A much easier way to see what is going on in this problem is to attempt a graphical representation (see Fig. 6).

The isoquants of the given firm are linear:

$$f(x_1, x_2) = c \Leftrightarrow \sqrt{2x_1 + 4x_2} = c \Leftrightarrow x_2 = \frac{c^2}{4} - \frac{x_1}{2}$$

Putting them together with an isocost line $\quad x_2 = \dfrac{C}{w_2} - \dfrac{w_1}{w_2} x_1$

we get the following:

- If $2w_1 < w_2$, the isocost is flatter than the isoquant. In order to produce any given level of output $\bar{y}$, the least cost way is to operate on the lower right corner, employing only input 1.
- If $2w_1 > w_2$, the isocost is steeper than the isoquant. In order to produce any given level of output $\bar{y}$, the least cost way is to operate on the upper left corner, employing only input 2.
- If $2w_1 = w_2$, the isocost is parallel to the isoquant. In order to produce any given level of output $\bar{y}$, any combination of the two inputs that lies on the given isoquant is optimal.

Finally, the demand for inputs is given[5]:

$$(x_1, x_2) = \begin{cases} \left(\dfrac{1}{2}\left(\dfrac{w_1}{p}\right)^2 - 2x, x \right) & if \quad 2w_1 = w_2 \\[2em] \left(0, \left(\dfrac{p}{w_2}\right)^2 \right) & if \quad 2w_1 > w_2 \\[2em] \left(\dfrac{1}{2}\left(\dfrac{p}{w_1}\right)^2, 0 \right) & if \quad 2w_1 < w_2 \end{cases}$$

Recall that $y = \sqrt{2x_1 + 4x_2}$. Substituting the optimal combination of inputs into the objective, we get the profit function:

$$\pi(p, w_1, w_2) = \begin{cases} w_1 - \dfrac{w_1^3}{2p^2} & if \quad 2w_1 = w_2 \\[2em] \dfrac{p^2}{w_2} & if \quad 2w_1 > w_2 \\[2em] \dfrac{p^2}{2w_1} & if \quad 2w_1 < w_2 \end{cases}$$

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[5] Note the indeterminacy of the input demand when $2w_1 = w_2$. This arises from the very fact that the isoquants are straight lines in this problem (i.e. the two inputs are perfect substitutes in the production of output).

7.   (FALSE)   Consider Darlene's optimization problem

$$\max_{(x_1,x_2,x_3)} U(x_1,x_2,x_3) = x_1^4 x_2^7 x_3$$

s.t.

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \le w \quad x_1, x_2, x_3 \ge 0$$

The Lagrangean:
$$L(x_1,x_2,x_3;\lambda,\mu_1,\mu_2,\mu_3) = x_1^4 x_2^7 x_3 + \lambda[w - p_1 x_1 - p_2 x_2 - p_3 x_3]$$
$$+\mu_1(x_1 - 0) + \mu_2(x_2 - 0) + \mu_3(x_3 - 0)$$

The first-order conditions:

Set (I)

$$\frac{\partial U(\underline{x})}{\partial x_1} = \lambda p_1 - \mu_1 \quad \Leftrightarrow \quad \lambda p_1 - \mu_1 = 4x_1^3 x_2^7 x_3 \qquad (1.1)$$

$$\frac{\partial U(\underline{x})}{\partial x_2} = \lambda p_2 - \mu_2 \quad \Leftrightarrow \quad \lambda p_2 - \mu_2 = 7x_1^4 x_2^6 x_3 \qquad (1.2)$$

$$\frac{\partial U(\underline{x})}{\partial x_3} = \lambda p_3 - \mu_3 \quad \Leftrightarrow \quad \lambda p_3 - \mu_3 = x_1^4 x_2^7 \qquad (1.3)$$

Set (II)

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \le w \qquad\qquad (2.1)$$
$$\lambda \ge 0 \qquad\qquad (2.2)$$
$$\lambda[w - p_1 x_1 - p_2 x_2 - p_3 x_3] = 0 \qquad\qquad (2.3)$$

$x_1 \ge 0$	$\mu_1 \ge 0$	$\mu_1 x_1 = 0$	(3.1)
$x_2 \ge 0$	$\mu_2 \ge 0$	$\mu_2 x_2 = 0$	(3.2)
$x_3 \ge 0$	$\mu_3 \ge 0$	$\mu_3 x_3 = 0$	(3.3)

First, notice that none of x_1, x_2, x_3 could be zero at an optimal point since, then, we would get zero overall utility (and, definitely, we could do better than that).
Hence, $x_1, x_2, x_3 > 0 \Rightarrow \mu_1 = \mu_2 = \mu_3 = 0$

Clearly, then, by any of the equations (1.1)-(1.3), $\lambda > 0$.

$$\frac{(1.1)}{(1.2)} \to \frac{4x_2}{7x_1} = \frac{p_1}{p_2} \quad \text{and} \quad \frac{(1.1)}{(1.3)} \to \frac{4x_3}{x_1} = \frac{p_1}{p_3} \quad \text{and} \quad p_1x_1 + p_2x_2 + p_3x_3 = w$$

The demand for good 2 will, therefore, be: $x_2 = \dfrac{7w}{12p_2}$

Clearly, if income doubles, the demand for good 2 exactly doubles.
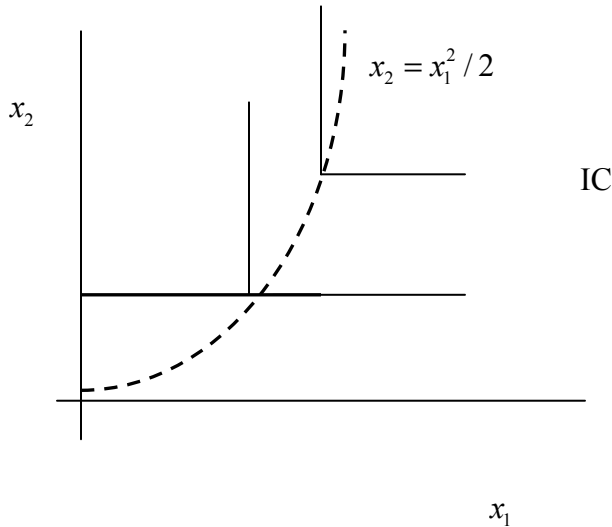

8.   (TRUE) For the given preferences, the optimal bundles will be given by the intersection of the curve $x_1^2 = 2x_2$ with the budget line. We solve:

$x_1^2 = 2x_2$ and $p_1x_1 + p_2x_2 = w$

If $p_1 = 0, p_2 > 0$ the slope of the budget line in a (x_1, x_2)-space will be $\dfrac{dx_2}{dx_1} = -\dfrac{p_1}{p_2} = 0$

(see Fig.8). Hence,

$$(x_1, x_2) = \left(x_1, \frac{w}{p_2} \right) : x_1 \geq \sqrt{\frac{2w}{p_2}}$$

a_0^1

x_2

$x_2 = x_1^2 / 2$

IC

Fig. 8

x_1

9.   (TRUE) Consider Angela's optimization problem

$$\max_{(x_1, x_2)} U(x_1, x_2) = x_1 + \sqrt{x_2}$$

$s.t.$

$$p_1x_1 + p_2x_2 \leq w \quad x_1, x_2 \geq 0$$

The Lagrangean:

$$L(x_1, x_2, p_1, p_2; \lambda, \mu_1, \mu_2) = x_1 + \sqrt{x_2} + \lambda[w - p_1 x_1 - p_2 x_2] + \mu_1(x_1 - 0) + \mu_2(x_2 - 0)$$

The first-order conditions:

Set (I)

$$\frac{\partial U(\underline{x})}{\partial x_1} = \lambda p_1 - \mu_1 \quad \Leftrightarrow \quad \lambda p_1 - \mu_1 = 1 \qquad (1.1)$$

$$\frac{\partial U(\underline{x})}{\partial x_2} = \lambda p_2 - \mu_2 \quad \Leftrightarrow \quad \lambda p_2 - \mu_2 = \frac{1}{2\sqrt{x_2}} \qquad (1.2)$$

Set (II)

$$p_1 x_1 + p_2 x_2 \le w \qquad (2.1)$$
$$\lambda \ge 0 \qquad (2.2)$$
$$\lambda[w - p_1 x_1 - p_2 x_2] = 0 \qquad (2.3)$$

$$x_1 \ge 0 \qquad \mu_1 \ge 0 \qquad \mu_1 x_1 = 0 \qquad (3.1)$$

$$x_2 \ge 0 \qquad \mu_2 \ge 0 \qquad \mu_2 x_2 = 0 \qquad (3.2)$$

Consider first interior solutions i.e. $x_1, x_2 > 0$:

We have: $\mu_1 = \mu_2 = 0$

From (1.1): $\lambda = \dfrac{1}{p_1} > 0$

Hence, the solution will be given by:

$$\frac{(1.1)}{(1.2)} \rightarrow \sqrt{x_2} = \frac{p_1}{2p_2} \quad \text{and} \quad p_1 x_1 + p_2 x_2 = w$$

i.e.

$$(x_1, x_2) = \left(\frac{w}{p_1} - \frac{p_1}{4p_2}, \left(\frac{p_1}{2p_2} \right)^2 \right)$$

Let now $x_1 > 0, \quad x_2 = 0$

We have: $\mu_1 = 0$

From (1.1): $\lambda = \dfrac{1}{p_1} > 0$

But, then, (1.2) cannot be satisfied with $x_2 = 0$ for finite, non-negative values of μ_2.
Hence, we cannot have a solution in this case.

Let now $x_1 = 0, \quad x_2 > 0$

We have: $\mu_2 = 0$

Hence, the solution will be given by:

$x_1 = 0 \qquad$ and $\qquad p_1 x_1 + p_2 x_2 = w$

i.e. $(x_1, x_2) = \left(0, \dfrac{w}{p_2} \right)$

From (1.2): $\lambda = \sqrt{\dfrac{p_2}{4w}} > 0$

From (1.1): $\mu_1 = \sqrt{\dfrac{p_1^2 p_2}{4w}} > 0$

Hence, this is acceptable as a corner solution.

In order now to decide which solution to keep, we need to compare the respective utility levels.

For the corner solution $(x_1, x_2) = \left(0, \dfrac{w}{p_2} \right)$ we get:

$$U^C (x_1, x_2) = \sqrt{\dfrac{w}{p_2}}$$

For the interior solution $(x_1, x_2) = \left(\dfrac{w}{p_1} - \dfrac{p_1}{4p_2}, \left(\dfrac{p_1}{2p_2} \right)^2 \right)$ we get:

$$U^I (x_1, x_2) = \dfrac{w}{p_1} - \dfrac{p_1}{4p_2} + \dfrac{p_1}{2p_2} = \dfrac{w}{p_1} + \dfrac{p_1}{4p_2} = \dfrac{4wp_2 + p_1^2}{4p_1 p_2}$$

Consider

$$U^I(x_1, x_2) > U^C(x_1, x_2) \Leftrightarrow \frac{4wp_2 + p_1^2}{4p_1p_2} > \sqrt{\frac{w}{p_2}} \Leftrightarrow \left(\frac{4wp_2 + p_1^2}{4p_1p_2}\right)^2 > \frac{w}{p_2}$$

$$\Leftrightarrow \left(4wp_2 + p_1^2\right)^2 > 16p_1^2 p_2 w \Leftrightarrow \left(4wp_2 - p_1^2\right)^2 > 0$$

Hence, we have verified that the interior solution is the dominant one.

Finally, the demand is: $(x_1, x_2) = \left(\dfrac{w}{p_1} - \dfrac{p_1}{4p_2}, \left(\dfrac{p_1}{2p_2}\right)^2\right)$

Clearly, changes in income have no effect on the amount of good 2 chosen.

10.   Suppose a person has indeed a demand curve for some commodity, let us denote it by j, which slopes up at all prices. Let other being equal and consider increases in the price of the given commodity.
From that person's budget constraint:

$$\sum_{i \in \{1,\dots,N\}-j} p_i x_i + p_j x_j = w$$

Clearly, since x_j rises following an increase in p_j, in order not to violate this constraint,

the agent must be decreasing his expenditure on all other goods $\displaystyle\sum_{i \in \{1,\dots,N\}-j} p_i x_i$. He will

have to be doing so until he gets this term to reach zero. From this point onwards, however, further increases in p_j cannot be followed by increases in x_j unless the budget

constraint is violated.
Hence, it cannot be true that the demand slopes up at all level of prices[6].

11.   (TRUE) Consider the general utility maximization problem, where X is the general consumption set:

$$\max_{\underline{x}} U(\underline{x})$$

$$s.t.$$

$$\underline{x} \in B_{\underline{p},w} = \left\{\underline{x} \in X : \underline{p}\underline{x} \le w\right\}$$

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[6] Mathematically, we clam that the demand function $x_j(\underline{p}, w)$ cannot be upwards slopping for levels of

$p_j > \overline{p}_j$ (where $\overline{p}_j : x_j\left((p_{-j}, \overline{p}_j), w\right) = \dfrac{w}{\overline{p}_j}$ for any given p_{-j}).

Let $\tilde{x} \in X$ the solution to the above problem. We are given that, following a price change, say to $\underline{p}'$, the new optimal bundle is some $\underline{x}' \in X$ such that $U(\underline{x}') > U(\tilde{x})$. Clearly, $\underline{x}' \notin B_{\underline{p},w}$ otherwise $\tilde{x} \in X$ could not have been maximizing utility under the constraint $B_{\underline{p},w}$. But the very fact that $\underline{x}'$ was not feasible under the constraint $B_{\underline{p},w}$, means exactly that the new bundle $\underline{x}'$ costs more, at the old prices $\underline{p}$ than the old bundle $\tilde{x}$.

12.   By Roy's identity:

$$x_1(p_1, p_2, y) = -\frac{\dfrac{\partial V}{\partial p}}{\dfrac{\partial V}{\partial y}} = -\frac{-\dfrac{1}{p_1 + p_2}}{\dfrac{1}{y}} = \frac{y}{p_1 + p_2}$$

$$x_2(p_1, p_2, y) = \frac{y}{p_1 + p_2}$$

For an endowment of $(e_1, e_2) = (1, 0)$ we get $y = p_1$

The offer curve: $\underline{x}(p_1, p_2) = \left(\dfrac{p_1}{p_1 + p_2}, \dfrac{p_1}{p_1 + p_2} \right)$

Income Elasticity of Demand for good 1:

$$\varepsilon_{1,y}(p_1, p_2, y) = \frac{\dfrac{\partial x_1}{x_1}}{\dfrac{\partial y}{y}} = \frac{\partial x_1}{\partial y} \frac{y}{x_1} = \left(\frac{1}{p_1 + p_2} \right) \left(\frac{1}{p_1 + p_2} \right)^{-1} = 1$$

Own-Price Elasticity of Demand for good 1:

$$\varepsilon_{1,p_1}(p_1, p_2, y) = \frac{\dfrac{\partial x_1}{x_1}}{\dfrac{\partial p_1}{p_1}} = \frac{\partial x_1}{\partial p_1} \frac{p_1}{x_1} = -\frac{y}{(p_1 + p_2)^2} p_1 \left(\frac{y}{p_1 + p_2} \right)^{-1} = -\frac{p_1}{(p_1 + p_2)}$$

13.   **(a)**   Let p_1 the price of all other goods.
Plan A: $p_1 x_1 + 2x_2 \leq 100 - 20 = 80$
Plan B: $p_1 x_1 + x_2 \leq 100 - 40 = 60$

(b)   Consider the general problem

$$\max_{(x_1,x_2)} U(x_1,x_2) = x_1 x_2$$

$$s.t.$$

$$p_1 x_1 + p_2 x_2 \leq w \quad x_1, x_2 \geq 0$$

The Lagrangean:

$$L(x_1,x_2,p_1,p_2;\lambda,\mu_1,\mu_2) = x_1 x_2 + \lambda[w - p_1 x_1 - p_2 x_2] + \mu_1(x_1 - 0) + \mu_2(x_2 - 0)$$

The first-order conditions:

Set (I)

$$\frac{\partial U(\underline{x})}{\partial x_1} = \lambda p_1 - \mu_1 \quad \Leftrightarrow \quad \lambda p_1 - \mu_1 = x_2 \qquad (1.1)$$

$$\frac{\partial U(\underline{x})}{\partial x_2} = \lambda p_2 - \mu_2 \quad \Leftrightarrow \quad \lambda p_2 - \mu_2 = x_1 \qquad (1.2)$$

Set (II)

$$p_1 x_1 + p_2 x_2 \leq w \qquad\qquad\qquad (2.1)$$
$$\lambda \geq 0 \qquad\qquad\qquad (2.2)$$
$$\lambda[w - p_1 x_1 - p_2 x_2] = 0 \qquad\qquad\qquad (2.3)$$

$$x_1 \geq 0 \qquad \mu_1 \geq 0 \qquad \mu_1 x_1 = 0 \qquad (3.1)$$

$$x_2 \geq 0 \qquad \mu_2 \geq 0 \qquad \mu_2 x_2 = 0 \qquad (3.2)$$

Note, however, that $x_1, x_2 \neq 0$ at the optimal point, otherwise the agent gets zero utility. Hence, we can only get interior solutions i.e. $x_1, x_2 > 0$ and, by equations (3.1), (3.2): $\mu_1 = \mu_2 = 0$.

Moreover, since we are considering interior solutions, $MU_{x_i} = x_j > 0$ for at least one commodity i. Consequently, by its corresponding equation (1.i), we get $\lambda p_i = x_j > 0 \Rightarrow \lambda > 0$.

Thus, we are also on the budget line equation at the optimum.

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The solution will be given by:

$$\frac{(1.1)}{(1.2)} \rightarrow \frac{x_1}{x_2} = \frac{p_2}{p_1} \qquad \text{and} \qquad p_1 x_1 + p_2 x_2 = w$$

Finally: $(x_1, x_2) = \left(\dfrac{w}{2p_1}, \dfrac{w}{2p_2} \right)$

Plan A: $w = 80, \quad p_2 = 2$. Hence:

$$x_1 = \frac{40}{p_1}, \quad x_2 = 20 \qquad U(x_1, x_2) = U\left(\frac{40}{p_1}, 20 \right) = \frac{800}{p_1}$$

Plan B: $w = 60, \quad p_2 = 1$. Hence:

$$x_1 = \frac{30}{p_1}, \quad x_2 = 30 \qquad U(x_1, x_2) = U\left(\frac{30}{p_1}, 30 \right) = \frac{900}{p_1}$$

(c) Clearly, we will opt for plan B.

14.   The output is given:

$$y = \frac{\partial \pi}{\partial p_1} = \frac{2p_1}{p_2 + p_3}$$

Similarly, the inputs are given:

$$x_2 = \frac{\partial \pi}{\partial p_2} = -\left(\frac{p_1^2}{p_2 + p_3} \right)^2 \qquad \text{and} \qquad x_3 = \frac{\partial \pi}{\partial p_3} = -\left(\frac{p_1^2}{p_2 + p_3} \right)^2$$

Own-Price Elasticity of Output:

$$\varepsilon_{y,p_1}(p_1, p_2, p_3) = \frac{\partial y}{\partial p_1} \frac{p_1}{y} = \frac{2}{(p_2 + p_3)} \left(\frac{2}{p_2 + p_3} \right)^{-1} = 1$$

15.   By Roy's identity:

$$x(p,y) = -\frac{\dfrac{\partial V}{\partial p}}{\dfrac{\partial V}{\partial y}} = -\frac{\dfrac{-b'(p)-(y-b(p))a'(p)}{a(p)}}{\dfrac{1}{a(p)}} = b'(p)+(y-b(p))a'(p)$$

The Engel curves are given by the above Marshallian demand function by fixing the price and allowing only income to be variable (obviously, different values for the fixed parameter $\bar{p}$ will correspond to different curves).

Hence, they will be the form: $x(\bar{p},y) = b'(\bar{p})+(y-b(\bar{p}))a'(\bar{p})$

They are linear in the single variable y.

16.   Let us assume that apples is the jth commodity out of a total of N commodities in the economy. The Apple problem:

$$\min_{(x_1,\dots,x_N)} \sum_{i=1}^{N} p_i x_i$$

$$s.t.$$

$$U(x_1,\dots,x_N) \geq u$$

$$\sum_{i=1}^{N} p_i x_i = p_j a_j$$

This is, of course, the expenditure minimization problem with the additional constraint that the money spent on a consumption bundle must be equal to the value of one's endowment in apples a_j.

Consider again the expenditure minimization problem alone

$$\min_{(x_1,\dots,x_N)} \sum_{i=1}^{N} p_i x_i$$

$$s.t.$$

$$U(x_1,\dots,x_N) \geq u$$

Let us denote by $\underline{x}^*(p_1,\dots,p_N,u)$ the solution to this problem. The value function is our

familiar expenditure function: $e(p_1,\dots,p_N,u) = \sum_{i=1}^{N} p_i x_i^*(p_1,\dots,p_N,u)$

The additional constraint that the Apple problem examines will give for the required level of apples-endowment the following:

$$e(p_1,...,p_N,u) = p_j a_j^*$$

$$\Leftrightarrow a_j^*(p_1,...,p_N,u) = \frac{1}{p_j}e(p_1,...,p_N,u)$$

$$\Leftrightarrow a_j^*(p_1,...,p_N,u) = \sum_{i\neq j}\frac{p_i}{p_j}x_i^* + x_j^*$$

Hence, the apple function is of the form $a_j^* = \underline{\tilde{p}}\underline{x}^*$

where $\underline{\tilde{p}} = \left(\frac{p_1}{p_j},...,\frac{p_{j+1}}{p_j},1,\frac{p_{j+1}}{p_j},...,\frac{p_N}{p_j}\right) = \frac{1}{p_j}\underline{p}$.

Concave in Prices
Reproduce the proof of concavity in prices for the expenditure function (JR pp.37-38) for the apple function $a_j^* = \underline{\tilde{p}}\underline{x}^*$.

Homogeneity of Degree Zero in Prices:
This is obvious. If we scale up all prices by a factor $\lambda > 0$, the problem becomes:

$$\min_{(x_1,...,x_N)} \sum_{i=1}^{N}\lambda p_i x_i$$
$$s.t.$$
$$U(x_1,...,x_N) \geq u$$
$$\sum_{i=1}^{N}\lambda p_i x_i = \lambda p_j e_j$$

The constraints remain the same while we are now considering an increasing transformation of the objective. The optimal solution $\underline{x}^*$ to the problem will remain the same as before. Of course, the new expenditure level required to purchase the bundle $\underline{x}^*$ will be scaled up now to $\lambda\underline{p}\underline{x}^*$, yet so will be the amount of income provided by the endowment $\lambda p_j e_j^*$. Hence, the required amount of apples-endowment will remain the same as before.

Non-decreasing in the Price of Every Good except Apples
Again, this is identical to the proof of the corresponding property for the expenditure function (JR pp.38). Consider the Lagrangean of the Apple problem:
$$L(\underline{x};\lambda,\mu) = \underline{p}\underline{x} + \lambda[U(\underline{x})-u] + \mu(\underline{p}\underline{x} - p_j a_j)$$

By the Envelope theorem:

$$\frac{\partial e(\underline{p},u)}{\partial p_i} = \frac{\partial L(\underline{x}^*,u)}{\partial p_i} = (1+\mu)x_i^* \geq 0 \qquad \text{for any } i \neq j$$

But here

$$\frac{\partial e(\underline{p},u)}{\partial p_i} = \frac{\partial(\underline{p}\underline{x}^*)}{\partial p_i} = \frac{\partial(p_j a_j^*)}{\partial p_i} = p_j \frac{\partial a_j^*}{\partial p_i}$$

i.e.

$$\frac{\partial a_j^*}{\partial p_i} = (1+\mu)\frac{x_i^*}{p_j} \geq 0 \qquad \text{for any } i \neq j$$

<u>Decreasing in the Price of Apples</u>
By the Envelope theorem:

$$\frac{\partial e(\underline{p},u)}{\partial p_j} = \frac{\partial L(\underline{x}^*,u)}{\partial p_j} = (1+\mu)x_j^* - a_j^* \qquad \text{for any } i \neq j$$

But now

$$\frac{\partial e(\underline{p},u)}{\partial p_j} = \frac{\partial(p_j a_j^*)}{\partial p_j} = a_j^* + \frac{\partial a_j^*}{\partial p_j}$$

Hence

$$\frac{\partial a_j^*}{\partial p_j} = (1+\mu)x_j^* - 2a_j^* \iff$$

$$\frac{\partial a_j^*}{\partial p_j} = (1+\mu)x_j^* - 2\left(\sum_{i \neq j}\frac{p_i}{p_j}x_i^* + x_j^*\right) = -(1-\mu)x_j^* - 2\sum_{i \neq j}\frac{p_i}{p_j}x_i^*$$

But by the Lagrangean first order conditions, we get[7]:

$$\frac{\partial L(\underline{x}^*,u;\lambda,\mu)}{\partial x_j} = (1+\mu)x_j^* + \lambda\frac{\partial U(\underline{x}^*)}{\partial x_j} = 0 \iff (1+\mu)x_j^* = -\lambda\frac{\partial U(\underline{x}^*)}{\partial x_j} < 0$$

Therefore, we conclude that $\dfrac{\partial a_j^*}{\partial p_j} < 0$.

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[7] I am assuming that the marginal utility with respect to apples is strictly positive at $\underline{x}^*$. Note that this assumption suffices to guarantee that $\lambda > 0$ (i.e. utility is driven down to $U(\underline{x}^*) = u$ at the optimum).

17.   In matrix notation, the problem can be written as follows:

$$\max_{x_1,x_2} \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

$$\begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq c_1$$

$$\begin{pmatrix} a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq c_2$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

or equivalently

$$\max_{x_1,x_2} \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

The Lagrangean:

$$L\left(x_{11}, x_{21}, x_{12}, x_{22}; \lambda_1, \lambda_2, \mu_1, \mu_2\right) = b_1 x_1 + b_2 x_2$$
$$+ \lambda_1 \left(c_1 - a_{11}x_1 - a_{12}x_2\right) + \lambda_2 \left(c_2 - a_{21}x_1 - a_{22}x_2\right) + \mu_1 \left(x_1 - 0\right) + \mu_2 \left(x_2 - 0\right)$$

The dual problem

$$\min_{x_1,x_2} \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

18.

$$\max_{(x_{11},x_{21})} U^1\left(x_{11},x_{21}\right)$$

$$s.t.$$

$$U^2\left(x_{12},x_{22}\right) \geq U^2\left(\omega_{12},\omega_{22}\right)$$

$$x_{11}+x_{12}=\omega_{11}+\omega_{12}$$

$$x_{21}+x_{22}=\omega_{21}+\omega_{22}$$

The Lagrangean:

$$L\left(x_{11},x_{21},x_{12},x_{22};\lambda,\mu_1,\mu_2\right)=U^1\left(x_{11},x_{21}\right)+\lambda\left[U^2\left(x_{12},x_{22}\right)-U^2\left(\omega_{12},\omega_{22}\right)\right]$$

$$+\mu_1\left(x_{11}+x_{12}-\omega_{11}-\omega_{12}\right)+\mu_2\left(x_{21}+x_{22}-\omega_{21}-\omega_{22}\right)$$

<u>Interpreting the Lagrangean multiplier:</u>

The value of the Lagrangean multiplier, at the optimal point, is equal to the marginal utility of player 1 with respect to the parameter of the corresponding constraint.

For example, at the solution, $\lambda^* = \dfrac{\partial U^1\left(x_{11}^*,x_{12}^*\right)}{\partial\left(-U^2\left(\omega_{21},\omega_{22}\right)\right)} = -\dfrac{\partial U^1\left(x_{11}^*,x_{12}^*\right)}{\partial U^2\left(\omega_{21},\omega_{22}\right)}$ gives the rate of

change of the objective function (here the utility of the first agent) with respect to infinitesimal decreases in the required level of utility that we specify as the lower permissible bound for player 2.

Similarly, $\mu_1^* = \dfrac{\partial U^1\left(x_{11}^*,x_{12}^*\right)}{\partial\left(\omega_{11}+\omega_{21}\right)}$ and $\mu_2^* = \dfrac{\partial U^1\left(x_{11}^*,x_{12}^*\right)}{\partial\left(\omega_{12}+\omega_{22}\right)}$

19.   Let the production of apples be modeled by the production function: $y_a = f\left(L_1\right)$

Similarly, let the production of breadfruit denoted by: $y_b = F\left(L_2,x_a\right)$

In equilibrium, all markets must clear. For the market for apples, this means that the breadfruit producer must be employing the entire output of the apple producer. Similarly, for the market for labor services, it means that, between them, the two producers must be employing the entire amount of labor L_s supplied by consumers.

We can, thus, re-write the equilibrium-level production function of breadfruits as:

$$y_b = F\left(L_2,y_a\right)=F\left(L_2,f\left(L_1\right)\right)=F\left(L_2,f\left(L_s-L_2\right)\right)$$

In this format, the problem seems very much like the Robinson Crusoe one. Consider the analysis of the Robinson-Crusoe problem in JR (pp.215). A shift in preferences away from leisure and towards breadfruits means that the new equilibrium allocation for the consumers would be a shift towards the northwest direction in Fig.5.8(b). Hence, the

breadfruits producer ought to be also moving his tangency point in a north-west direction in Fig.5.8(a) i.e. his new is-profit line should be flatter. We should expect, therefore, the relative price of labor to breadfruits (the slope of the iso-profit and budget lines in Fig.5.8.) to fall.

As for the price of apples, this would be determined by the profit-maximizing condition for the production of apples:

$$p_a f'(L_1) = w \Leftrightarrow \frac{w}{p_a} = f'(L_1)$$

Since the apples producer will be absorbing some of the increase in labor supply and the production function for apples is increasing, we should expect the relative price of labor to apples to be larger in the equilibrium.

Hence, in the new equilibrium, apples become cheaper than before relatively to both other commodities and labor becomes cheaper relatively to breadfruits.

20.   The given function

$$V(p_1, p_2, y) = \frac{y}{p_1 + 2\sqrt{p_1 p_2} + p_2}$$

cannot be an indirect utility function because it is not homogenous of degree zero in prices.
For any $\lambda > 0$,

$$V(\lambda p_1, \lambda p_2, y) = \frac{y}{\lambda p_1 + 2\sqrt{\lambda^2 p_1 p_2} + \lambda p_2} = \frac{V(p_1, p_2, y)}{\lambda} \neq V(p_1, p_2, y)$$