Problem Set #5

Suggested Solutions

1. Consider the utility contour of agent h i.e. the set of consumption bundles $x^h = \left(a^h, b^h \right)$ that give the same utility level $\bar{u}$. This will be given:

$$\left\{ x^h : u^h \left(x^h \right) = \bar{u} \right\}$$

Since the utility remains constant at the given level $\bar{u}$ along the contour, if we totally differentiate the utility function with respect to the vector x^h, we will get:

$$du^h \left(x^h \right) = d\bar{u} = 0 \Leftrightarrow$$

$$\frac{\partial u^h}{\partial a^h} da^h + \frac{\partial u^h}{\partial b^h} db^h = 0 \Leftrightarrow$$

$$\frac{db^h}{da^h} = -\frac{\dfrac{\partial u^h}{\partial a^h}}{\dfrac{\partial u^h}{\partial b^h}}$$

This is the slope of the utility contour, at the given point x^h, in a $\left(a^h, b^h \right)$ space. We call the negative of this slope the ***Marginal Rate of Substitution*** (MRS) between a^h and b^h at the point x^h.

$$MRS^h_{a,b} = -\frac{db^h}{da^h}\bigg|_{u=\bar{u}} = \frac{\dfrac{\partial u^h}{\partial a^h}}{\dfrac{\partial u^h}{\partial b^h}} = \frac{MU^h_a}{MU^h_b} \quad \text{(I)}$$

Note that this gives, by definition, the rate at which agent h is ***willing*** to trade between the goods a and b in order to maintain his level of utility constant at $\bar{u}$. Clearly, from (I), he can do so by exchanging 1 unit of good b for $\dfrac{MU^h_a}{MU^h_b}$ units of a.

In the given problem, for $h = 1$, we get: $MRS_{a,b}^1 = -\dfrac{db^1}{da^1}\bigg|_{u=\bar{u}} = \dfrac{\dfrac{\partial u^1}{\partial a^1}}{\dfrac{\partial u^1}{\partial b^1}} = \dfrac{MU_a^1}{MU_b^1} = \dfrac{1}{\left(b^1\right)^{-1}} = b^1$

Taking the agent's consumption set to be $X^1 = R_+^2$, we can draw this MRS as an

horizontal line in the $\left(a^1, b^1\right)$ space - i.e. it is constant for a given value of b^1,

independent of the value of a^1 (see Fig. I).

Similarly, for $h = 2$, we get: $MRS_{a,b}^2 = -\dfrac{db^2}{da^2}\bigg|_{u=\bar{u}} = \dfrac{\dfrac{\partial u^2}{\partial a^2}}{\dfrac{\partial u^2}{\partial b^2}} = \dfrac{MU_a^2}{MU_b^2} = \left(a^2\right)^{-1}$

Taking this agent's consumption set to be also $X^1 = R_+^2$, we can draw his MRS as in Fig.

II. Note that this is independent of the value of b^2 and, hence, constant along a vertical

line in the $\left(a^2, b^2\right)$ space for a given value of a^2.
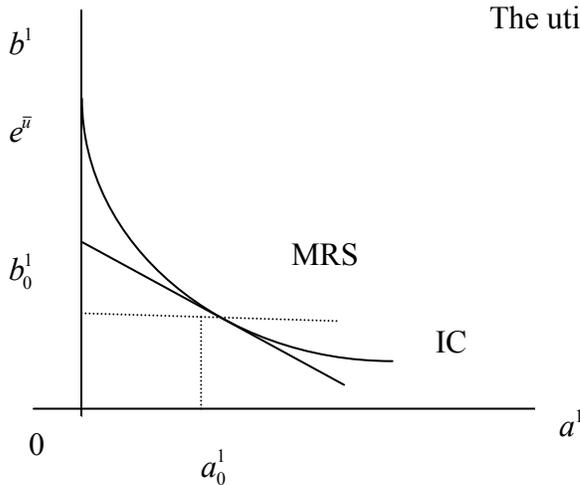
The utility contours are given:

$$a^1 + \log b^1 = \bar{u} \quad \Leftrightarrow \quad b^1 = e^{\bar{u}-a^1} = Ae^{-a^1}$$
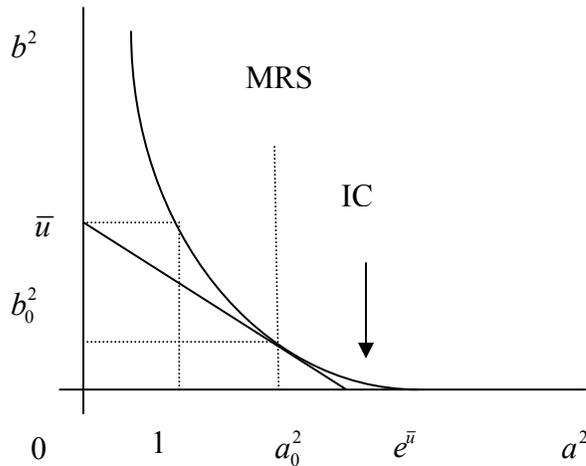

Fig. I

Note:

At the given point $\left(a_0^1, b_0^1\right)$, the indifference curve going through

the point (and corresponding to the utility level $\bar{u}$) is shown as IC.

Its slope - i.e. the slope of its tangent at the point $\left(a_0^1, b_0^1\right)$ - is equal

to $-b_0^1$. Moreover, if we move along the depicted IC, the slope of the contour changes according to the equation

$$\left.\frac{db^1}{da^1}\right|_{u=\bar{u}} = -MRS^1 = -b^1$$

The utility contours are given:

$$\log a^2 + b^2 = \bar{u} \iff b^2 = -\log a^2 + \bar{u}$$

Fig. II

Note:

At the given point $\left(a_0^2, b_0^2\right)$, the indifference curve going through the point (and corresponding to the utility level $\bar{u}$) is shown as IC. Its slope is equal to $-1/a_0^2$. Moreover, if we move along the depicted IC, the slope of the contour changes according to the

equation $\left.\dfrac{db^2}{da^2}\right|_{u=\bar{u}} = -MRS^2 = -\left(a^2\right)^{-1}$

Consider now an economy with the two goods *a* and *b* and the two agents whose preferences we just described. Imagine that we flip Fig. II by 180-degrees clockwise and place on top of Fig. I. The result would be the Edgeworth box of Fig. III.
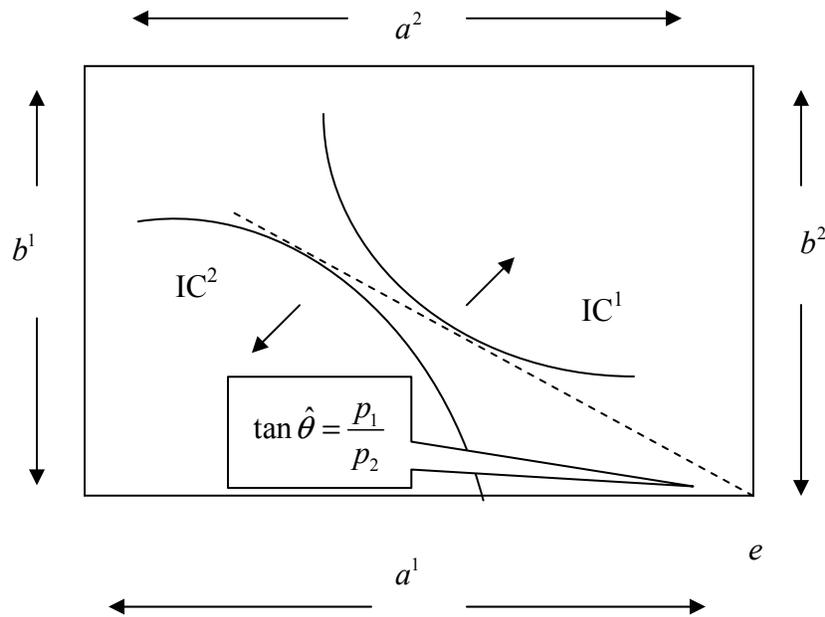
Fig. III

Digression on the vary basics of Walrasian Equilibrium Analysis

Consider an exchange economy with H consumers, indexed by $h \in \{1,...,H\}$, and N commodities, indexed by $i \in \{1,...,N\}$. Let us start with a given endowment e, depicted by an $N \times H$ matrix where the element of the i-th row and h-th column will be denoted by $\left[e_{i,m} \right]_{N \times H} = e_i^h$ and taken to mean the quantity of the i-th commodity that the h-th agent is endowed with.

In this economy, a ***Walrasian Equilibrium*** is a price vector $\underset{(N \times 1)}{p} = \left[p_i \right]_{i=1}^{N}$ and an allocation matrix $\left[\hat{x}_{i,m} \right]_{N \times H} = \hat{x}_i^h$ such that:

1. For each agent, taking the price vector as given, the *h*-th column of $\left[\hat{x}_{i,m}\right]_{N\times H}$ - depicted by the vector $\hat{x}^h_{(N\times 1)} = \left[\hat{x}^h_i\right]^N_{i=1}$ - solves the following utility maximization problem:

$$\max_{x^h} U^h\left(x^h\right)$$

$$\text{s.t. } px^h \le pe^h \iff \sum_{i=1}^N p_i x_i^h \le \sum_{i=1}^N p_i e_i^h$$

2. The allocation is feasible under exchange of the given endowment. In other words:

$$x = e \iff x_i = e_i \qquad \forall i \in \{1,...,N\}$$

$$\iff \sum_{h=1}^H x_i^h = \sum_{h=1}^H e_i^h \quad \forall i \in \{1,...,N\}$$

Consider now agent *h* solving his optimization problem.

The Lagrangean function can be written as:

$$L\left(x_1^h,...,x_N^h;\lambda,\mu_1,...,\mu_N\right) = U^h\left(x_1^h,...,x_N^h\right) + \lambda^h\left(\sum_{i=1}^N p_i e_i^h - \sum_{i=1}^N p_i x_i^h\right) + \mu_1^h\left(x_1^h - 0\right) + ... + \mu_N^h\left(x_N^h - 0\right)$$

The first order conditions would be:

Set (I)

$$\frac{\partial L}{\partial x_i^h} = 0 \iff \frac{\partial U^h\left(x^h\right)}{\partial x_i^h} = \lambda^h p_i - \mu_i^h \qquad \forall i \in \{1,...,N\}\,(1)$$

Set (II)

$$\frac{\partial L}{\partial \lambda^h} \ge 0 \iff \sum_{i=1}^H p_i x_i^h \le \sum_{i=1}^H p_i e_i^h \qquad\qquad (2.1)$$

$$\lambda^h \ge 0 \qquad\qquad (2.2)$$

$$\lambda^h \frac{\partial L}{\partial \lambda^h} = 0 \iff \lambda^h \left[\sum_{i=1}^{H} p_i e_i^h - \sum_{i=1}^{H} p_i x_i^h \right] = 0 \qquad (2.3)$$

$$\frac{\partial L}{\partial \mu_i^h} \geq 0 \iff x_i^h \geq 0 \qquad \forall i \in \{1,...,N\} \qquad (3.1)$$

$$\mu_i^h \geq 0 \qquad \forall i \in \{1,...,N\} \qquad (3.2)$$

$$\mu_i^h \frac{\partial L}{\partial \mu_i^h} = 0 \iff \mu_i^h x_i^h = 0 \qquad \forall i \in \{1,...,N\} \qquad (3.3)$$

Let us assume that (a) we will consider only interior solutions for this optimization problem (i.e. that the agent chooses to consume $\hat{x}_i^h > 0 \quad \forall i \in \{1,...,N\}$) and (b) the agent is not satiated with respect to at least one commodity (i.e. there exist commodity, say

$j \in \{1,...,N\}$, for which $MU_j^h = \dfrac{\partial U^h}{\partial x_j} > 0 \quad \forall x_j \in R_+$).

Then, it is clear from assumption (a) that $\mu_i^h = 0 \quad \forall i \in \{1,...,N\}$.

Hence, the N equations embedded in (1) become:

$$\frac{\partial U^h(x^h)}{\partial x_i^h} = \lambda^h p_i \qquad \forall i \in \{1,...,N\} \qquad (1.i)$$

Moreover, by assumption (b), if the price vector is strictly positive (i.e. $p >> 0 \iff p_i > 0 \quad \forall i \in \{1,...,N\}$), we conclude that $\lambda^h > 0$ at the optimal point.

Consider now equation (1) across two different commodities. Let us take commodities i and j. We have:

$$\left. \begin{array}{l} \dfrac{\partial U^h(x^h)}{\partial x_i^h} = \lambda^h p_i \\[2em] \dfrac{\partial U^h(x^h)}{\partial x_j^h} = \lambda^h p_j \end{array} \right\} \Rightarrow \dfrac{\dfrac{\partial U^h(x^h)}{\partial x_i^h}}{\dfrac{\partial U^h(x^h)}{\partial x_j^h}} = \dfrac{p_i}{p_j} \iff MRS_{i,j}^h = \dfrac{p_i}{p_j} \qquad (I)$$

Let us now return to our economy as a whole. Looking at the optimization problem of each of the H agents, it becomes obvious, once we notice that all agents face the same

price vector p, that all agents' marginal rates of substitution must be the same across a given pair of commodities. In other words,

$$MRS_{i,j}^h = \frac{p_i}{p_j} = MRS_{i,j}^{\tilde{h}}$$

for any two agents $h, \tilde{h} \in \{1, ..., H\}$ and any two commodities $i, j \in \{1, ..., N\}$.

The Pareto Efficient Allocations

In general, an economic outcome is Pareto optimal (or Pareto efficient (PE)) if there is no alternative feasible outcome that makes every individual in the economy at least as well off and some individual strictly beter off.

In terms of the Edgeworth box, two-agent, pure exchange setting, we say that an allocation x is PE if there is no other allocation $\tilde{x}$ in the box (i.e. feasible) with $\tilde{x}^h \succeq_h x^h$ for $h = 1, 2$ and $\tilde{x}^h \succ_h x^h$ for some h.

Mathematically, the set of PE allocations in the Edgeworth box are given by the solution set of the following problem:

$$\max_{x_1^h, x_2^h, x_1^{h'}, x_2^{h'}} U^h\left(x_1^h, x_2^h\right)$$

$$s.t.$$

$$U^{h'}\left(x_1^{h'}, x_2^{h'}\right) \geq u$$

$$x_1^{h'} + x_1^h = e_1^{h'} + e_1^h = e_1$$

$$x_2^{h'} + x_2^h = e_2^{h'} + e_2^h = e_2$$

By varying the parameters $e_1^h, e_2^h, e_1^{h'}, e_2^{h'}, u$ of the problem, we can generate the entire set of PE allocations (the Pareto set (PS)).

Note that by setting up the Lagrangean of this problem and examining the first order conditions, it is easy to see that, in the *interior* of the box, if an allocation is PE then the MRS's of the two consumers must be equal at that point (i.e. the two indifference curves must be *tangent*).

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By the first fundamental theorem of welfare economics,

- $WEAs \subset PS$

The Contract Curve

The contract curve is defined as that part of the Pareto set where *both* consumers do at least as well as their *initial endowments*.

Note that whereas the Pareto set depends only on the agents' preferences (i.e. the shape of their indifference curves) the contract curve depends also on the particular endowment point that they start from (i.e. different initial endowment points for the same consumers will result to different contract curves).

- $Contract \quad Curve \subset PS$

The reasoning behind examining the concept of the contract curve is that we might expect any bargaining between the two agents to result in an agreement to trade to some point on the contract curve since this is the set of points at which both of them do at least as well as their endowments and for which there is no alternative trade that can make both of them better off.

Note also that the definition of the contract curve given in the textbook (see JR, pp. 183) is lacking because it applies *only* to allocations in the interior of the box. As we will show below, an allocation on some of the borders of the box can be on the contract curve without the indifference curves of the two agents being tangent to one another at that point.

The Core

A **coalition** is any non-empty subset of the set of consumers $H = \{1,...,H\}$ in an economy.

A coalition $S \subset H$ **improves upon** or **blocks** the feasible allocation $\tilde{x} = \left(\tilde{x}^1,...,\tilde{x}^H\right)$ if for every $h \in S$ we can find a consumption bundle $x^h \geq 0$ such that:

1     $x^h \succ_h \tilde{x}^h \quad \forall h \in S$

2     $\sum_{h \in S} x_i^h = \sum_{h \in S} e_i^h \quad \forall i \in \{1,...,N\}$

Note that this definition says that a coalition S can improve upon a feasible allocation $\tilde{x}$ if there is some way, by using <u>only</u> the endowments of <u>its members</u>, for the coalition to come with an aggregate bundle $x^S = \left(x^s\right)_{s \in S}$ to distribute amongst its members so as to make each one better off.

A feasible allocation $\tilde{x} = \left(\tilde{x}^1,...,\tilde{x}^H\right)$ has the **core property** if there is no coalition of agents $S \subset H$ that blocks it. The **core** is the set of allocations that have the core property.

- $WEAs \subset Core$

When we have only two consumers $H = \{1,2\}$ the core <u>coincides</u> with the contract curve. With two consumers there can only be three possible coalitions: $\{1\},\{2\},\{1,2\}$. Any allocations that is not PE will be blocked by the grand coalition $\{1,2\}$. Any allocation in the Pareto set but not on the contract curve will be blocked by either $\{1\}$ or $\{2\}$ (an agent who is not at least as well off as at his endowment point can simply use his endowment as the blocking alternative).

With more than two agents, there can also be other blocking collations. However, the fact that the whole economy will be the grand coalition means that, in general, <u>*all allocations in the core must be PE*</u>.

- $Core \subset PS$

We could make a similar argument seeking a general relation between the contract curve and the core. The fact that each agent on his own can always form a coalition, using his endowment as a blocking bundle, means that, again in general, an allocation that is not on the contract curve cannot be in the core (since it would be blocked by whoever was not at least as well off as at his endowment point).

This statement reads mathematically: $\tilde{x} \notin Contract\ Curve \Rightarrow \tilde{x} \notin Core$

Taking the contrapositive statement: $\tilde{x} \in Core \Rightarrow \tilde{x} \in Contract\ Curve$

- $Core \subset Contract\ Curve$

Finally, putting all together:

- $WEAs \subset Core \subset Contract\ Curve \subset PS$

1.(Continued) For the given problem at hand, we have $H = 2,\ i \in \{a, b\}$.

At the interior of the Edgeworth box, each agent gets allocated strictly positive quantities of both goods. Hence, we are looking at an interior solution for both agents. Moreover, you should verify that assumption (b) is met for both commodities for both agents. Therefore, at an optimal allocation, both agents would be setting their respective marginal rates of substitution between the commodities *a* and *b* equal to their relative price:

$$MRS_{a,b}^1 = \frac{p_a}{p_b} = MRS_{a,b}^2 \iff \left(b^1\right)^{-1} = a^2 \iff a^2 b^1 = 1 \qquad \text{(II)}$$

However, in the interior of the Edgeworth box, the quantity allocated to each agent of each commodity being strictly positive means that it is also strictly less than the total endowment of the commodity (i.e. strictly less than 1 unit in this case)[1]. But $a^2, b^1 < 1$

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[1] Clearly, every agent is getting some amount of a given commodity, no one could be getting everything of that commodity.

cannot satisfy equation (II). Therefore, points in the interior of the box cannot lie on the contract curve.

Let us now examine the rest of the space of the Edgeworth box – that is, the four border segments.

It is obvious that agent 1 will never choose to consume zero of commodity *b* since this would give him infinitely negative utility. Similarly, agent 2 would never choose to consume zero of commodity *a*. Thus, points on the south and east border segments are out of the question. The same holds, of course, for the southwest, northwest and southeast corners.

The northwest corner is the allocation $\left(\hat{x}^1\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\left(\hat{x}^2\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is clearly feasible.

One would be tempted to argue that this allocation also seems to satisfy (II), therefore, it must be on the contract curve. This is an incorrect argument because it does not take into account the fact that, in order to derive (II), we employed the assumption (a) about interior solutions and this allocation does not involve strictly positive quantities of both goods to both agents.

To examine, therefore, the validity of this allocation point as a member of the contract curve, we have to go back and re-examine the optimization problem of each agent.

<u>For agent 1</u>

The first order conditions:

<u>Set (I)</u>

$$\frac{\partial U^1\left(x^1\right)}{\partial a^1} = \lambda^1 p_1 - \mu_1^1 \quad \Leftrightarrow \quad \lambda^1 p_1 - \mu_1^1 = 1 \qquad (1.1.\text{i})$$

$$\frac{\partial U^1\left(x^1\right)}{\partial b^1} = \lambda^1 p_2 - \mu_2^1 \quad \Leftrightarrow \quad \lambda^1 p_2 - \mu_2^1 = \frac{1}{b^1} \qquad (1.2.\text{i})$$

<u>Set (II)</u>

$$p_1 a^1 + p_2 b^1 \leq p_1 \qquad\qquad (2.1.\text{i})$$

$$\lambda^1 \geq 0 \qquad\qquad\qquad\qquad (2.2.\text{i})$$

$$\lambda^1 \left[p_1 - p_1 a^1 - p_2 b^1 \right] = 0 \qquad\qquad (2.3.\text{i})$$

$$a^1 \geq 0 \qquad\qquad \mu_1^1 \geq 0 \qquad\qquad \mu_1^1 a^1 = 0 \qquad (3.\text{a}.\text{i})$$

$$b^1 \geq 0 \qquad\qquad \mu_2^1 \geq 0 \qquad\qquad \mu_2^1 b^1 = 0 \qquad (3.\text{b}.\text{i})$$

The consumption bundle $\left(\hat{x}^1 \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives: $b^1 > 0 \ \Rightarrow \ \mu_2^1 = 0$.

From (1.2.i): $\lambda^1 p_2 = 1 \Leftrightarrow \lambda^1 = \dfrac{1}{p_2} > 0$

From (1.1.i): $\mu_1^1 = \dfrac{p_1}{p_2} - 1$

Note that, for $\mu_1^1 \geq 0$, we need $p_1 \geq p_2$. This later condition ensures also that (2.1.i) is satisfied. However, since $\lambda^1 > 0$, we need to satisfy (2.1.i) with equality, otherwise we would fail (2.3.i). Hence, we really need $p_1 = p_2$.

We conclude, therefore, that if $p_1 = p_2$, then $\left(\hat{x}^1 \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a solution to agent's 1

optimization problem.

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For agent 2

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The first order conditions:

<u>Set (I)</u>

$$\frac{\partial U^2 \left(x^2 \right)}{\partial a^2} = \lambda^2 p_1 - \mu_1^2 \ \Leftrightarrow \ \lambda^2 p_1 - \mu_1^2 = \frac{1}{a^2} \qquad (1.1.\text{ii})$$

$$\frac{\partial U^2 \left(x^2 \right)}{\partial b^2} = \lambda^2 p_2 - \mu_2^2 \ \Leftrightarrow \ \lambda^2 p_2 - \mu_2^2 = 1 \qquad (1.2.\text{ii})$$

<u>Set (II)</u>

$$p_1 a^2 + p_2 b^2 \leq p_2 \qquad\qquad\qquad (2.1.\text{ii})$$

$$\lambda^2 \geq 0 \qquad\qquad\qquad\qquad (2.2.\text{ii})$$

$$\lambda^2 \left[p_2 - p_1 a^2 - p_2 b^2 \right] = 0 \qquad\qquad (2.3.\text{ii})$$

$$a^2 \geq 0 \qquad\qquad \mu_1^2 \geq 0 \qquad\qquad \mu_1^2 a^2 = 0 \qquad (3.\text{a.ii})$$

$$b^2 \geq 0 \qquad\qquad \mu_2^2 \geq 0 \qquad\qquad \mu_2^2 b^2 = 0 \qquad (3.\text{b.ii})$$

The consumption bundle $\left(\hat{x}^2 \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives: $a^2 > 0 \;\Rightarrow\; \mu_1^2 = 0$.

From (1.1.ii): $\lambda^2 = \dfrac{1}{p_1} > 0$

From (1.2.ii): $\mu_2^2 = \dfrac{p_2}{p_1} - 1$

Note that, for $\mu_2^2 \geq 0$, we need $p_2 \geq p_1$. This later condition ensures also that (2.1.ii) is satisfied. However, since $\lambda^2 > 0$, we need to satisfy (2.1.ii) with equality, otherwise we would fail (2.3.ii). Hence, we really need $p_1 = p_2$.

We conclude, therefore, that if $p_1 = p_2$, then $\left(\hat{x}^2 \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is also solution to agent's 2

optimization problem.

Consequently, for $p_1 = p_2$, the north-west corner allocation $\left(\hat{x}^1 \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\left(\hat{x}^2 \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a

WEA. Therefore, it lies on the contract curve.

The Western Border

Let us now consider allocations on the western borderline of the Edgeworth box. That is points of the form $\left(\hat{x}^1 \right) = \begin{pmatrix} a^1 \\ b^1 \end{pmatrix}$, $\left(\hat{x}^2 \right) = \begin{pmatrix} a^2 \\ b^2 \end{pmatrix}$ with $a^1 = 0$, $a^2 = 1$, $b^1, b^2 \in (0,1)$

By inspection of the agents' optimization problems we see that we have:

For agent 1, the consumption bundle $\left(\hat{x}^1 \right) = \begin{pmatrix} 0 \\ b^1 \end{pmatrix} : b^1 \in (0,1)$ gives: $b^1 > 0 \;\Rightarrow\; \mu_2^1 = 0$.

From (1.2.i): $\lambda^1 = \dfrac{1}{p_2 b^1} > 0$

From (1.1.i): $\mu_1^1 = \dfrac{p_1}{p_2 b^1} - 1$

Note that, for $\mu_1^1 \geq 0$, we need $p_1 \geq p_2 b^1$. This later condition ensures also that (2.1.i) is satisfied. However, since $\lambda^1 > 0$, we need to satisfy (2.1.i) with equality, otherwise we would fail (2.3.i). Hence, we really need $p_1 = p_2 b^1 \Leftrightarrow b^1 = \dfrac{p_1}{p_2}$.

We conclude, therefore, that $\left(\hat{x}^1 \right) = \begin{pmatrix} 0 \\ \dfrac{p_1}{p_2} \end{pmatrix} : p_1 < p_2$ is a solution to agent's 1 optimization problem that lies on the western border segment of the box.

For agent 2, the consumption bundle $\left(\hat{x}^2 \right) = \begin{pmatrix} 1 \\ b^2 \end{pmatrix} : b^2 \in (0,1)$ gives: $\mu_1^2 = \mu_2^2 = 0$.

From (1.2.ii): $\lambda^2 = \dfrac{1}{p_2} > 0$

From (1.2.i): $\lambda^2 = \dfrac{1}{p_1}$

Hence, we require $p_1 = p_2$. But this condition disagrees with (2.1.ii) since it results in $p_2 b^2 = 0$ - where I have taken also into account the fact that, since $\lambda^2 > 0$, we need to satisfy (2.2.i) with equality, in the face of (2.3.ii).

We conclude, therefore, that $\left(\hat{x}^2 \right) = \begin{pmatrix} 1 \\ b^2 \end{pmatrix} : b^2 \in (0,1)$ cannot be a solution to agent's 2 optimization problem.

Note, however, that this point still offers the agent a positive utility level. It is, therefore, preferred to his endowment point.

Hence, the allocation $\left(\hat{x}^1 \right) = \begin{pmatrix} 0 \\ \dfrac{p_1}{p_2} \end{pmatrix}$, $\left(\hat{x}^2 \right) = \begin{pmatrix} 1 \\ \dfrac{p_2 - p_1}{p_2} \end{pmatrix} : p_1 < p_2$ is feasible and preferred by both agents to their respective endowments.

It lies, thus, on the contract curve but is not a WEA because it is not optimal for agent II.

The Northern Border

You should repeat the argument given above for the points on the northern border line of the Edgeworth box. That is allocations of the form $(\hat{x}^1) = \begin{pmatrix} a^1 \\ b^1 \end{pmatrix}$, $(\hat{x}^2) = \begin{pmatrix} a^2 \\ b^2 \end{pmatrix}$ with

$b^1 = 1,\ b^2 = 0,\ a^1, a^2 \in (0,1)$.

By the symmetry of the agents preferences and endowments, you should be able to see immediately that the allocations $(\hat{x}^1) = \begin{pmatrix} \dfrac{p_1 - p_2}{p_1} \\ 1 \end{pmatrix}$, $(\hat{x}^2) = \begin{pmatrix} \dfrac{p_2}{p_1} \\ 0 \end{pmatrix} : p_1 > p_2$ is feasible and preferred by both agents to their respective endowments. Thus, they lie on the contract curve.

In these cases, it is agent 2 who is maximizing utility. Agent 1 is not, although he does prefer the offered allocation to his endowment. Hence, these points are not WEA's[2].

The Core

Obviously, by the very way we construct the Edgeworth box, every point in the interior or on the boundary of the box, represents a feasible allocation. As already noted, in a two-person economy, we can only have two types of coalitions: (1) a coalition formed by one agent alone and (2) a coalition formed by both agents together. For this reason, in a two-consumer economy, the contract curve coincides with the core.

To see this, recall that we found three kinds of points:

(1) The north-west corner where both agents were maximizing their utility (hence, no-one has incentive to block it). Putting the same argument differently, this point was found to be a WEA and, hence, must be in the core.

(2) Points on the western border where agent 1 is maximizing utility while agent 2 gets something which is strictly preferred to his endowment. Here, one could be tempted into arguing that agent 2 wants to block every single point on this border segment because he would always prefer to move, for example, to a point directly below it in the same segment. To see this, note that for any allocation that gives agent 2

$(\hat{x}^2) = \begin{pmatrix} 1 \\ b^2 \end{pmatrix} : b^2 \in (0,1)$, agent 2 prefers to move directly below on the vertical direction

of the western segment since this would give him the same quantity of good a but more of good b. In principle, agent 2 would block every point of the western segment apart from the south-west corner which is not on the contract curve anyway.

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[2] The western and northern border points illustrate the reasoning for using the term contract curve. The points on the western border are not optimal for agent II. Yet if we were to allow for bargaining to take place prior to trading and consider a situation where agent I had all the bargaining power and was making take-it-or-leave-it offers to agent II, agent II would accept any point on this border because it leaves better off than his endowment. Similarly, points on the northern border would be accepted by agent I, if agent II was making take-it-or-leave-it to agent I.

This reasoning is flawed, however, because it ignores the requirement in the definition of blocking allocations that the alternative bundle that a coalition offers itself as a blocking device must be generated by the aggregate endowment of the coalition itself (i.e. by re-distributing across coalition members the total endowment of the coalition and of the coalition only). When the coalition consists of a single agent, the only blocking allocation available is his own endowment (since there is no one else in the coalition to switch endowments with).

In our case, therefore, although agent 2 would like to move at points vertically below any given point on the we stern border, he _cannot_ use them as blocking allocations because he cannot offer them to himself using _only_ his endowment.

(3) Similarly, points on the northern segment of the box could not be blocked by agent 1. For any given such point, agent 1 wants to move along the segment and to the right since such a change would give him the same quantity of good *b* (i.e. the total endowment) but more of good *a*. This incentive applies for all points on the segment apart from the northeast corner, which, nevertheless, is not a member of the contract curve. Yet, none of these points can be obtained from agent 1's endowment point. Consequently, they cannot be used as blocking coalitions.

We conclude, therefore, that the core is the same as the contract curve:

$$Core = Contract\ Curve = \left\{ \left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 1 \\ 1-x \end{pmatrix} \right) : x \in (0,1] \right\} \cup \left\{ \left(\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} 1-x \\ 0 \end{pmatrix} \right) : x \in (0,1) \right\}$$

(B) Let us now return to agent 1's optimization problem. The necessary and sufficient conditions for an optimal point are given by the set of equations (1.1.i)-(3.b.i). If one replaces the RHS of (2.1.i) with a general variable *y* for income and follows the same analysis that we did earlier in part (A), the Marshallian demand for agent 1 comes out to be:

$$\hat{x}^1 = \begin{pmatrix} \dfrac{y}{p_1}-1 \\ \dfrac{p_1}{p_2} \end{pmatrix} \qquad \text{for } p_1 \leq y$$

If we fix the income level *y* and we allow for only the price vector $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ to be a variable, we get the **offer curve**. Moreover, at the endowment point $e^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have $y = p_1$ and, thus, the agent demands nothing of good *a*. His offer curve, therefore,

becomes $\tilde{x}^1 = \begin{pmatrix} 0 \\ \dfrac{p_1}{p_2} \end{pmatrix}$ which is the vertical axis that goes through the given endowment point.

Similarly, for agent 2, we get his Marshallian demand:

$$\hat{x}^2 = \begin{pmatrix} \dfrac{p_2}{p_1} \\ \dfrac{y}{p_1} - 1 \end{pmatrix} \text{ for } p_1 \leq y$$

At the endowment point $e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have $y = p_2$ and, thus, the agent demands nothing

of good b. His offer curve, therefore, becomes $\tilde{x}^2 = \begin{pmatrix} \dfrac{p_2}{p_1} \\ 0 \end{pmatrix}$ which is the horizontal axis

that goes through the given endowment point.

The two offer curves intersect at the point $\left(x^1, x^2\right) = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ for $p_1 = p_2$. We have

already shown that this point, at prices $p_1 = p_2$, is a Walrasian Equilibrium allocation (WEA).

(C) Note first of all that changes in the endowment of agent 1 would correspond to opposite-direction changes in the endowment of agent 2.

For the Contract Curve: Recall our result that points in the interior of the box cannot lie on the contract curve (irrespective of the endowment point, since the derivation of the result did not use the endowment point at all).

Hence, changes in the initial endowment of agent 1 (and accordingly of agent 2) could only possibly affect the original right-angles contract curve set (recall that this was the union of the northern and western borders, excluding the upper-right and lower-left corners). One could think that the contract curve might shrink due to the fact that, by endowing initially agent 1 with some positive amount of good b, his initial utility level is no longer negative infinity. Therefore, it should be harder now to maintain points on the borders as preferred to an endowment point. The same reasoning could be also applied

from agent 2's perspective to claim that endowing him with some positive amount of good *a* could shrink the contract curve.

This reasoning is not correct though. Consider, for example, points on the northern border. Previously, although not optimal for agent 1, they were acceptable because they were infinitely better than his endowment (see Fig. IV). Any such given point (say *A* in Fig. IV), though, will remain in the contract curve if we move the endowment point from *e* to any other point in the segment *eA* since:

    (a) *A* would still offer agent 1 higher utility than at his new endowment point

    (b) maintaining the same price ratio as originally at *A* (i.e. using again the slope of the line *eA* as the price ratio), agent 2 continues to be optimizing at *A*.

The same argument can be applied to show that any point on the western frontier can continue to be supported as being on the contract curve (see Fig. V).

The WEA can be maintained by any endowment point on the diagonal of the box.

Finally, any point on the eastern and southern borders (including the northeast and southwest corners) continues not to be on the contract curve since it would involve some agent getting zero amount of the good that enters his utility in log-form. Consequently, some agent would derive infinitely negative utility from this allocation and he would be desperate to drop against any endowment point other than the original southeast corner.

a^1 ⟷ e **Fig. IV**

IC^1

b^1

IC^2

a^1 ⟷ e **Fig. V**
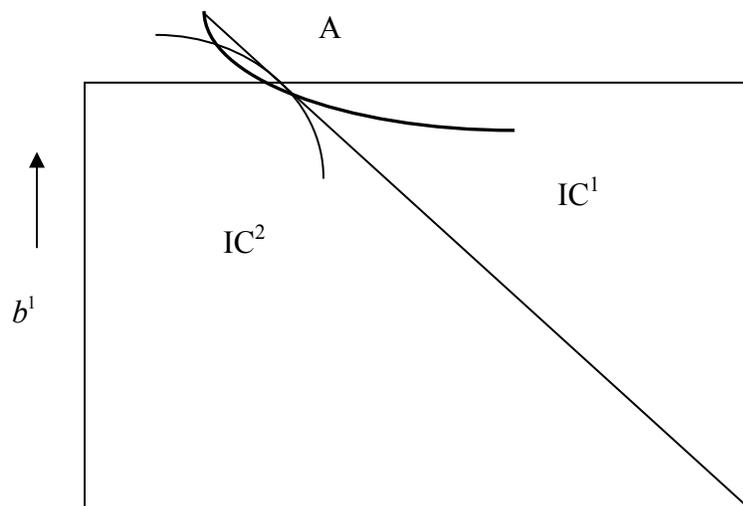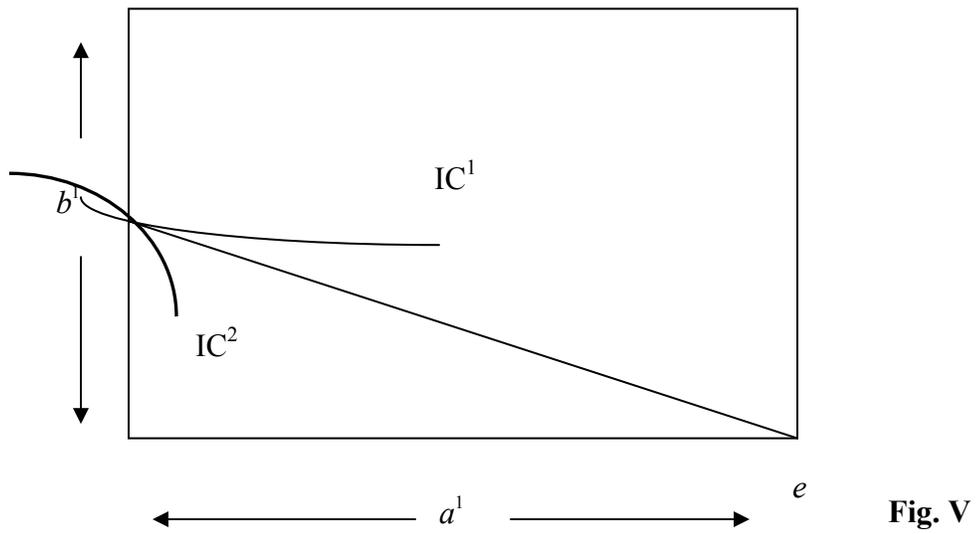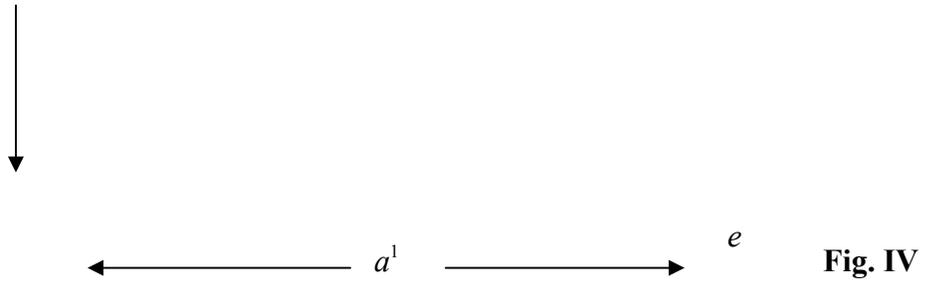
For the Offer Curves: In part (B), we found that, in general, the offer curve of agent 1 is given:

$$\hat{x}^1 = \begin{pmatrix} \dfrac{y}{p_1} - 1 \\[2mm] \dfrac{p_1}{p_2} \end{pmatrix} \quad \text{for } p_1 \le y$$

Substituting for $y = p_1 e_a^1 + p_2 e_b^1$ we get $\hat{x}^1 = \begin{pmatrix} e_a^1 + \dfrac{p_2}{p_1} e_b^1 - 1 \\[2mm] \dfrac{p_1}{p_2} \end{pmatrix}$

Similar, for agent 2:

$$\hat{x}^2 = \begin{pmatrix} \dfrac{p_2}{p_1} \\ \dfrac{y}{p_1} - 1 \end{pmatrix} \quad \text{for } p_1 \leq y \quad \text{i.e. } \hat{x}^2 = \begin{pmatrix} \dfrac{p_2}{p_1} \\ e_a^2 + \dfrac{p_2}{p_1} e_b^2 - 1 \end{pmatrix}$$

Hence, we see that, changes in the endowment of agent 1 and (correspondingly) agent 2, other things being equal, would leave unaffected the demands for agent 1 and 2 for goods *b* and *a* respectively.

However, they do affect agent 1's demand for good *a* and 2's demand for good *b*. The offer curves for these two commodities are linear in the price ratio with the endowments determining their slopes and their intercepts. Reducing the endowment of good *a* for agent 1 (and, thus, increasing the endowment of agent 2 in this good) will result in a downward shift of the offer curve of agent 1 for good *a* and an upward one for the curve of agent 2 for good *b*. Similarly, increasing the endowment of good *b* for agent 1 (and, thus, decreasing the endowment of agent 2 in this good) will result in making the offer curve of agent 1 for good *a* steeper while flattening the curve of agent 2 for good *b*.

JR 5.12

(a)   Consider each agent's utility maximization problem.

For agent 1, we know that, since his preferences are described by

$$U^1(x_1, x_2) = \min(x_1, x_2)$$

his optimal bundle will be given by the intersection of the line

$$x_1 = x_2$$

and his budget line

$$p_1 x_1 + p_2 x_2 = p_1 e_1 + p_2 e_2 \,^3$$

Given his endowment point $e^1 = (30, 0)$ we get:

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[3] Of course, the actual budget constraint is $p_1 x_1 + p_2 x_2 \leq p_1 e_1 + p_2 e_2$. However, this agent is non-satiated in either of the two goods. Hence, he will clearly spend all of his income and choose a consumption bundle that lies on the budget line.

$$x_1 = x_2$$

$$p_1 x_1^1 + p_2 x_2^1 = p_1 e_1^1 + p_2 e_2^1 = 30 p_1 \qquad \text{i.e.} \qquad \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} \dfrac{30 p_1}{p_1 + p_2} \\ \dfrac{30 p_1}{p_1 + p_2} \end{pmatrix} \qquad \text{(I)}$$

For agent 2, we are given the indirect utility function. Using Roy's identity:

$$\begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} -\dfrac{\dfrac{\partial V^2}{\partial p_1}}{\dfrac{\partial V^2}{\partial y}} \\[2em] -\dfrac{\dfrac{\partial V^2}{\partial p_2}}{\dfrac{\partial V^2}{\partial y}} \end{pmatrix} = \begin{pmatrix} \dfrac{y}{2 p_1} \\[1.5em] \dfrac{y}{2 p_2} \end{pmatrix} = \begin{pmatrix} \dfrac{p_1 e_1^2 + p_2 e_2^2}{2 p_1} \\[1.5em] \dfrac{p_1 e_1^2 + p_2 e_2^2}{2 p_2} \end{pmatrix}$$

Given his endowment point $e^2 = (0, 20)$ we get:
$$\begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} \dfrac{20 p_2}{2 p_1} \\[1.5em] \dfrac{20 p_2}{2 p_2} \end{pmatrix} = \begin{pmatrix} \dfrac{10 p_2}{p_1} \\[1em] 10 \end{pmatrix} \qquad \text{(II)}$$

Equations (I) and (II) define the optimal allocations for the two agents. To determine the WEA's, we need to keep only the feasible ones out of the set that (I) and (II) define.

Feasibility for good 1:

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 = 30 \Leftrightarrow$$
$$\frac{30 p_1}{p_1 + p_2} + \frac{10 p_2}{p_1} = 30 \Leftrightarrow \qquad (1)$$
$$\frac{3 p_1}{p_1 + p_2} + \frac{p_2}{p_1} = 3$$

Feasibility for good 2:

$$x_2^1 + x_2^2 = e_2^1 + e_2^2 = 20 \Leftrightarrow$$

$$\frac{30p_1}{p_1 + p_2} + 10 = 20 \Leftrightarrow \qquad (2) \qquad \text{Note that } p_2 = 2p_1 \text{ satisfies } (1)$$

$$2p_1 = p_2$$

Hence, the WEs are: $\left\{ (x^1, x^2, p) = \left(\begin{pmatrix} 15 \\ 15 \end{pmatrix}, \begin{pmatrix} 10 \\ 10 \end{pmatrix}, \begin{pmatrix} p \\ 2p \end{pmatrix} \right) \right\} : p > 0$

(b)   Consider the new endowment vector for agent 1: $e^1 = (5, 0)$

Reviewing his optimization problem, we see that his demand is now given:

$$\begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} \dfrac{5p_1}{p_1 + p_2} \\ \dfrac{5p_1}{p_1 + p_2} \end{pmatrix} \qquad (I.i)$$

Agent 2's demand is obviously still given by equation (II).

Let us now work out again the feasible allocations.

Feasibility for good 1:

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 = 5 \Leftrightarrow$$

$$\frac{5p_1}{p_1 + p_2} + \frac{10p_2}{p_1} = 5 \Leftrightarrow \qquad (1.i)$$

$$\frac{p_1}{p_1 + p_2} + \frac{2p_2}{p_1} = 1$$

Feasibility for good 2:

$$x_2^1 + x_2^2 = e_2^1 + e_2^2 = 20 \Leftrightarrow$$

$$\frac{5p_1}{p_1 + p_2} + 10 = 20 \Leftrightarrow \qquad (2.i) \qquad \text{But } p_1 = 2p_2 \text{ does not satisfies } (1.i)$$

$$p_1 = 2p_2$$

Therefore, when the endowment vector is $e^1 = (5, 0)$ for agent, a Walrasian equilibrium does not exist in this economy. The existence theorem fails because assumption (5.1) in

JR (pp. 188) is not met in this setting since the utility function of agent 1 is not strongly increasing on R_+^2.

JR 5.13

Recall our duality relations between the expenditure and indirect utility functions:

$$V(p,e(p,u))=u$$

$$e(p,u)=e(p,V(p,y))=y$$

We can, therefore, re-state the given problem as follows:

$$e^1(p,u)=e^1\left(p,V^1(p,y^1)\right)=y^1$$

$$\Leftrightarrow \sqrt[3]{3\left(\frac{3}{2}\right)^2 p_1^2 p_2 e^u}=y^1$$

$$\Leftrightarrow \sqrt[3]{3\left(\frac{3}{2}\right)^2 p_1^2 p_2 e^{V^1(p,y^1)}}=y^1$$

$$\Leftrightarrow V^1(p,y^1)=\ln\left(\frac{4}{27}\frac{(y^1)^3}{p_1^2 p_2}\right)$$

Similarly, for agent 2 (note the symmetry of the given expenditure functions):

$$V^2(p,y^2)=\ln\left(\left(\frac{4}{27}\right)\frac{(y^2)^3}{p_2^2 p_1}\right)$$

We can now apply Roy's identity, as in the previous question, to get the Marshallian demands of the two agents.

For agent 1:

$$\begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} -\dfrac{\dfrac{\partial V^1}{\partial p_1}}{\dfrac{\partial V^1}{\partial y^1}} \\ -\dfrac{\dfrac{\partial V^1}{\partial p_2}}{\dfrac{\partial V^2}{\partial y^1}} \end{pmatrix} = \begin{pmatrix} \dfrac{2y^1}{3p_1} \\ \dfrac{y^1}{3p_2} \end{pmatrix} = \begin{pmatrix} \dfrac{2\left(p_1 e_1^1 + p_2 e_2^1\right)}{3p_1} \\ \dfrac{p_1 e_1^1 + p_2 e_2^1}{3p_2} \end{pmatrix}$$

Given his endowment point $e^1 = (10,0)$ we get: $\qquad \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} \dfrac{20}{3} \\ \dfrac{10 p_1}{3 p_2} \end{pmatrix}$ (I)

For agent 2:

$$\begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} -\dfrac{\dfrac{\partial V^2}{\partial p_1}}{\dfrac{\partial V^2}{\partial y^2}} \\ -\dfrac{\dfrac{\partial V^2}{\partial p_2}}{\dfrac{\partial V^2}{\partial y^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{y^2}{3p_1} \\ \dfrac{2y^2}{3p_2} \end{pmatrix} = \begin{pmatrix} \dfrac{p_1 e_1^2 + p_2 e_2^2}{3p_1} \\ \dfrac{2\left(p_1 e_1^2 + p_2 e_2^2\right)}{3p_2} \end{pmatrix}$$

Given his endowment point $e^2 = (0,10)$ we get: $\qquad \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} \dfrac{10 p_2}{3 p_1} \\ \dfrac{20}{3} \end{pmatrix}$ (II)

Equations (I) and (II) define the optimal allocations for the two agents. To determine the WEA's, we need to keep only the feasible ones out of the set that (I) and (II) define.

Feasibility for good 1:

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 = 10 \Leftrightarrow \frac{20}{3} + \frac{10 p_2}{3 p_1} = 10 \Leftrightarrow p_2 = p_1 \qquad (1)$$

Feasibility for good 2:

$$x_2^1 + x_2^2 = e_2^1 + e_2^2 = 10 \Leftrightarrow \frac{20}{3} + \frac{10 p_1}{3 p_2} = 10 \Leftrightarrow p_1 = p_2 \qquad (2)$$

Hence, the WEs are: $\left\{ \left(x^1, x^2, p \right) = \left(\left(\begin{array}{c} \frac{20}{3} \\ \frac{10}{3} \end{array} \right), \left(\begin{array}{c} \frac{10}{3} \\ \frac{20}{3} \end{array} \right), \left(\begin{array}{c} p \\ p \end{array} \right) \right) \right\} : p > 0$
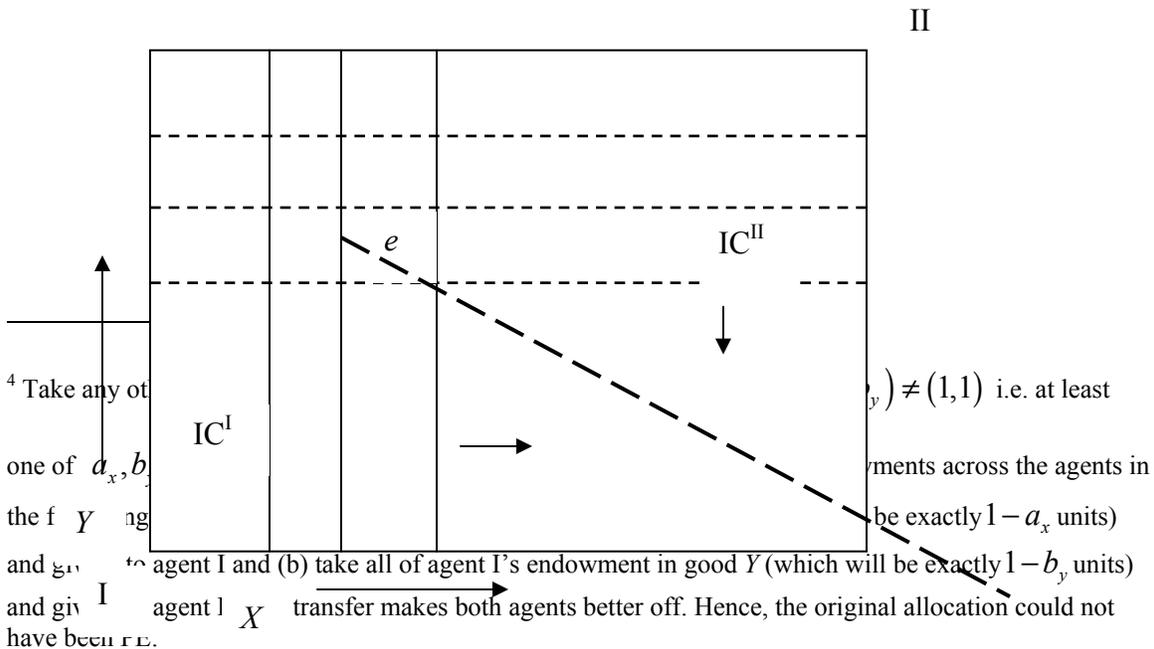
JR 5.1 (a)   The set of pareto efficient allocations in this economy consists of only the south-east corner of the box[4]. Yet, this point is not a WEA, for the given initial endowment shown by the point e in the graph and the (uniquely, in this case, associated) price vector depicted by the bold dotted line.

To see this, consider agent I's budget line $\left\{ \left(x^I, y^I \right) \in R_+^2 : p_x x^I + p_y y^I = p_x e_x^I + p_y e_y^I \right\}$

Denote by w^I the income of agent I and by e_x the total endowment of good x in the economy.

Clearly, if $\dfrac{w^I}{p_x} > e_x \Leftrightarrow p_x e_x^I + p_y e_y^I > p_x \left(e_x^I + e_x^{II} \right)$ (I)

his budget line extends all the way out to some point A which lies outside the box.

II

[4] Take any ot ... $,_y) \neq (1,1)$ i.e. at least

one of a_x, b ... ments across the agents in

the f Y ng ... be exactly $1 - a_x$ units)

and g_I ... to agent I and (b) take all of agent I's endowment in good Y (which will be exactly $1 - b_y$ units)

and giv ^{I} agent I X transfer makes both agents better off. Hence, the original allocation could not have been FE.

A

B

Fig. I

Given preferences for agent I as depicted by the vertical-lines indifference curves of Fig. I (i.e. corresponding to a utility function of the form $U(x,y) = x$), the agent will optimally choose to spend his entire income on good x. Thus, he will choose point A as his optimal bundle.
But this means that the south-east corner of the box cannot be a WEA as agent I is not at his optimal consumption point.

Of course, a similar argument could be made from agent II's perspective.

If $\quad \dfrac{w^{II}}{p_y} > e_y \Leftrightarrow p_x e_x^{II} + p_y e_y^{II} > p_y \left(e_y^{I} + e_y^{II} \right) \qquad$ (II)

then, his budget line extends all the way out to some point B which lies outside the box (in the same direction as A on the line from the endowment point).

Note, however, that one only of (I),(II) could ever hold[5]. Whichever agent it holds for, it is sufficient to establish the non-existence of WE here.

(b)

II

[5] It is terribly ... hould also verify that they cannot both ...

I   X   **Fig. II**
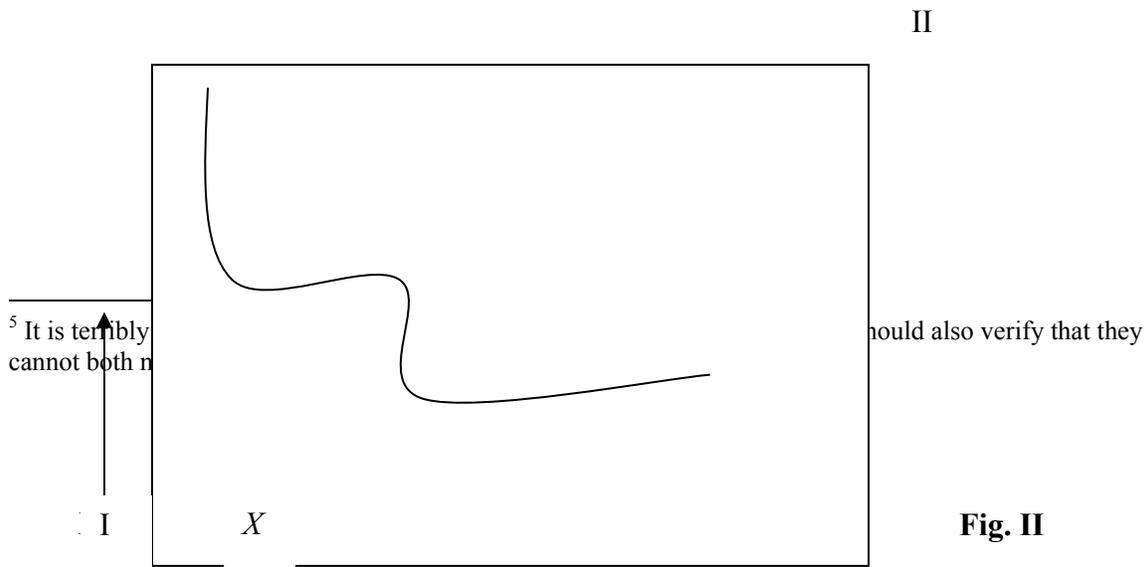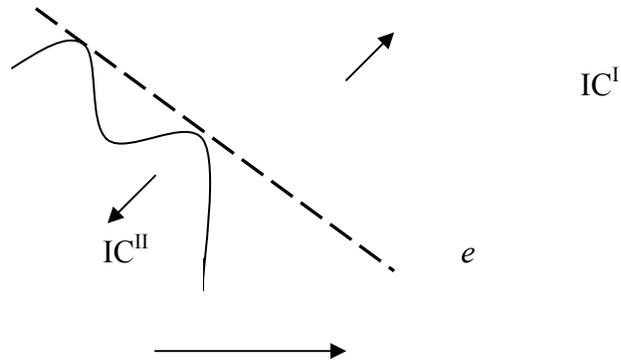
Digression on Continuity of Preferences

Definition of Continuity (version I)[6]:

For all $x \in X$, the upper contour set of x_0: $UC_{x_0} = \{x \in X : x \succeq x_0\}$ and the lower contour set: $LC_{x_0} = \{x \in X : x_0 \succeq x\}$ are closed in *X*.

The above is the general definition of continuity for preferences. However, often textbooks use also the following definition as equivalent:

Definition of Continuity (version II):

The preference relation $\succeq$ on X is ***continuous*** if it is preserved under limits. That is, for any sequence of pairs of bundles $\left\{ \left(x^n, y^n \right) \right\}_{n=1}^{\infty}$ with $x^n \succeq y^n \quad \forall n$ such that $x^n \xrightarrow{n \to \infty} x$ and $y^n \xrightarrow{n \to \infty} y$, we have $x \succeq y$.

Note that, strictly speaking, the two are not equivalent unless *X* is a metrizable space. Generally, for *X* being any topological space, the second version *implies* continuity under the first (i.e. the second version is sufficient for the general definition) but the sequence definition of continuity is valid *only* for metrizable spaces.

In most applications, we take the consumption space to be some subset of R^n which is metrizable (indeed we assume the standard topology using the Euclidean distance metric) and therefore we do not need to worry about the fine print here.

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[6] In JR (see axiom 3, pp. 8) the upper and lower contour sets are called "at least as good as" set and "no better than" set respectively. Moreover, the consumption set X is taken to be R_+^n. I am giving the general definition of continuity here but will also be using R_+^n as the consumption set shortly for graphical representations.
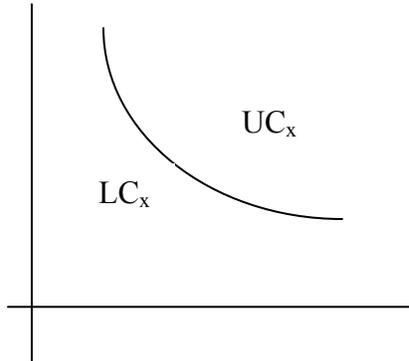
Example of Continuous Preferences (i)

Fig. 1

Let $X = R_+^2$ and consider and any point x on the indifference curve shown in Fig. 1. Assuming that the depicted preferences are strictly monotonic, the "better than x" ("no better than x") set is the entire subset of R_+^2 that lies above (below) the curve. These preferences being continuous means that the curve itself belongs to *both* of these sets. In other words, $UC_x \cap LC_x = IC_x$. The indifference is obviously the boundary of each contour set and the requirement that both sets be closed in R_+^2 implies that it should belong to both sets.
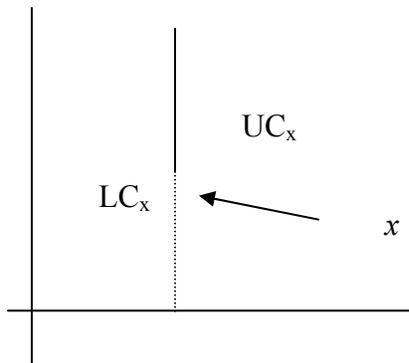
Example of Discontinuous Preferences

Fig. 2

Let again $X = R_+^2$. The standard example of a preference relation that is not continuous is the **Lexicographic** ordering that ranks bundles in the same way a dictionary is organized.

Define, thus, $x \succeq y \iff (x_1 = y_1) \vee \left(x_1 > y_1 \ \wedge \ x_2 > y_2 \right)$

In words, commodity 1 has the highest priority in determining the preference ordering just as the first letter word does in the ordering of a dictionary. When the consumption level of the first commodity is the same across two bundles, the amount of the second commodity determines then the preference ranking just as, in the case of the words having the same first letter, we compare the second letters for the dictionary ordering.

Consider the indifference sets for this preference relation. It is easy to see that they are singleton, point-sets. Every point x in R^2_+ forms a contour set (i.e. any other point in R^2_+ will be either better or worse than x). Fig. 2 depicts such a point x. The "better than x" consists of all those points in R^2_+ that lie to the right of the vertical axis that goes through the point x including only the solid segment of the axis. Algebraically, this is the set
$$\left\{ \tilde{x} \in R^2_+ : \left(\tilde{x}_1 > x_1 \ \wedge \ \tilde{x}_2 \in R_+ \right) \vee \left(\tilde{x}_1 = x_1 \ \wedge \ \tilde{x}_2 > x_2 \right) \right\}.$$
The "better than x" set is not closed in R^2_+ because any point in the dotted segment of the vertical axis that goes through x (i.e. any point in $\left\{ \tilde{x} \in R^2_+ : \tilde{x}_1 = x_1 \ \wedge \ \tilde{x}_2 < x_2 \right\}$) is a limit point of this set, yet it does belong in it.

Clearly, the "no better than x" set which consists of all points in R^2_+ that lie to the left of the vertical axis that goes through x including the dotted segment of the axis, is not closed either. The points that lie on the solid segment of the vertical axis that goes through x are limit points of the "no better than x" set but do not belong in the set itself.

Algebraically, the discontinuity of preferences can be easily shown here using version II of the definition of continuity given above.
Consider the following sequences of bundles:
$$\left\{ \left(x^n_1, x^n_2 \right) \right\}^\infty_{n=1} : x^n_1 = 1 + \frac{1}{n}, \ \ x^n_2 = 0 \qquad \text{and} \qquad \left\{ \left(y^n_1, y^n_2 \right) \right\}^\infty_{n=1} : y^n_1 = 1, \ \ y^n_2 = 1$$
We have:
$$\left(x^n_1, x^n_2 \right) \succ \left(y^n_1, y^n_2 \right) \quad \forall n \in Z_+ \qquad \left(x^n_1, x^n_2 \right) \overset{n \to \infty}{\to} (1,0) \qquad \text{but} \qquad (1,0) \prec (1,1)$$
i.e. the lexicographic preferences exhibit reversal of preference in the limit and, consequently, they are not continuous.
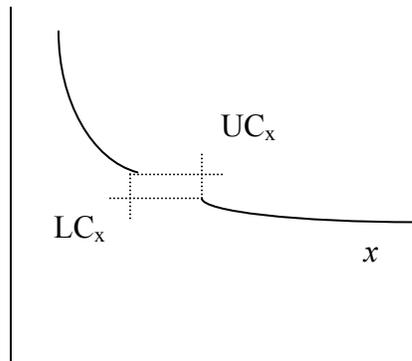
Example of Continuous Preferences 2

Fig. 3

29

_____

Consider the indifference curve shown in Fig.3 which is discontinuous over the region enclosed in the dotted box. Let us assume that preferences are strictly monotonic (i.e. other things being equal, the agent always strictly prefers having more of either good) but there regions (like the one depicted by the interior of the dotted box) over which the agent is unable to express preference.

Points on the dotted line segments of the no-decision box are clearly limit points of the "better than x" and "no better than x" sets[7]. But, by strict monotonicity, they are also members of the respective sets. Hence, both sets are closed and, therefore, the preferences of Fig. 3 are continuous.

JR 5.1(Continued)

   (c)   Consider a situation where both agents have Lexicographic prefferences with x being the dominant good in both. Then, both agents would wish to spend their entire endowment on purchsing good X (i.e. exchange all of their respective endowments of good Y with good X) and there can be no equilibrium trade between them since they both want the same thing. No WE would exist in this situation.

Consider now a sotuation where both agents have lexicographic prefrences but agent I views X as the dominant good while agent II places his priority on good Y. In this case, we get a situation very similar to that in Fig. I except for the fact that the indifference sets are now points in the box.

Fix now the endowment vector e such that it lies on the 45-degree from the lower-left corner of the Edgeworth box.

It is easy now to check that any price vector of the form $\left(p_x, p_y\right) = \left(p, p\right)$ for $p > 0$ gives:

$$\frac{w^h}{p} = e_i \Leftrightarrow pe_x^h + pe_y^h = p\left(e_i^I + e_i^{II}\right)$$

$$\text{for } \left\{h = I \wedge i = x\right\} \ \left\{h = II \wedge i = y\right\} \text{ respectively.}$$

Hence, with such prices, each agent wants to optimally choose the lower left point of the box and this point is a WEA.

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[7] Points on the northern and eastern dotted segments belong to the "better than x" set whereas points on the southern and western segments belong to the "no better than x" set.