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Final Exam Solutions

1. Consider the following simultaneous-move strategic-form game:

Player 2

Player 1		A	B	C
	U	(1,1)	(-1,0)	(-1,-1)
	D	(-1,-2)	(1,0)	(1,1)

Find all the Nash equilibria (pure and mixed) of this game.

Let u be the probability that 1 plays U, and a,c be the probabilities that player 2 plays A and C, respectively. Then, the expected payoffs are

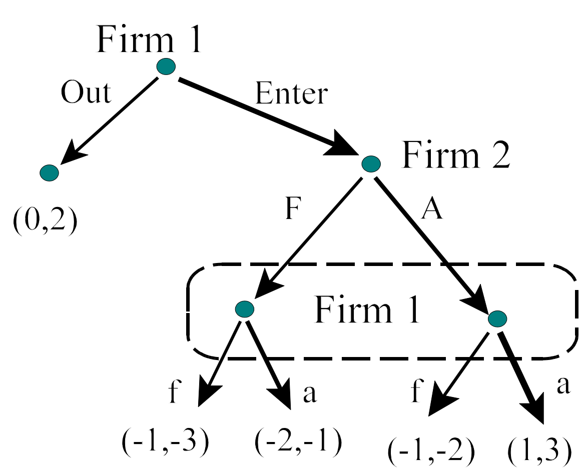
$$\pi_1 = (1-2a)(1-2u) \qquad \text{and} \qquad \pi_2 = (3u-2)a + (1-2u)c.$$

Firm 1 maximizes its response to a by u = 1 when a > ½, u = 0 when a < ½, and any u when a = ½, Firm 2 maximizes its response to u by a = 1 when u > 2/3, by c = 1 when u < ½, and by a= c = 0 when ½ < u < 2/3, by c = 0 and any a when u = 2/3, and by a = 0 and any c when u = ½. Then, u = a = 1 is one Nash equilibrium, u = a = 0 is another. The corresponding pure strategies are (U,A) and (D,C). Finally, a = ½, c = 0, and u = 2/3 is a mixed strategy Nash equilibrium.

2. Firm 1 has to decide whether to enter a market in which Firm 2 is the incumbent. If Firm 1 enters, then the firms have to decide simultaneously whether to fight (F or Firm 2/f for Firm 1) or accommodate (A for Firm 2/a for Firm 1), with the payoffs given in the graph below.

(A) Suppose there can be no communication between the firms before the game. Find all the subgame-perfect Nash equilibria, pure or mixed.

(B) Suppose Firm 2 can legally pre-commit itself to fight if Firm 1 enters, say by contracting with each of its customers that it will strictly beat any price posted by Firm 1. How does this change the game, and the subgame-perfect Nash equilibria?



Do backward induction. If entry, the firms play the following game in strategic form:

		Player 2	
		F	A
Player 1	f	(-1,-3)	(-1,-2)
	a	(-2,-1)	(1,3)

This game has a unique pure strategy Nash equilibrium at (a,A), so the extensive game can be reduced to a decision by Firm 1 to enter with payoff (1,3) or stay out with payoff (0,2). Then, entry and accommodation is the unique subgame-perfect Nash equilibrium. If the game is changed so that Firm 2 can pre-commit to F, then a is a dominant strategy in the final strategic form game, and Firm 1 faces the effective payoffs (-2,-1) with entry and (0,2) with no entry. Then, choosing no entry is a dominant strategy for Firm 1 in the reduced game.

3. Two firms F' and $F''$are said to be <u>Cournot duopolists</u> if they produce a common product in quantities q' and q'' at a marginal cost $m > 0$, and then face price $p = A - B(q' + q'')$ in the market, where $A > m$ and $B > 0$ are constants known to both firms.

(A) Find a Nash equilibrium when the two firms choose quantities simultaneously.

(B) How would the Nash equilibrium change if firm F' first chooses q', and then knowing this, firm F'' chooses q''? Which firm gains and which loses relative to (A)?

Firm F' has payoff $\pi' = (A - B(q' + q'') - m)q'$, maximized at $q' = (A-m)/2B - q''/2$. Symmetrically, firm 2 has the optimal reaction function $q''= (A-m)/2B - q'/2$. These two equations are solved at $q' = q'' = (A-m)/3B$, yielding price $(A+2m)/3$ and a profit of $(A-m)^2/9B$ to each firm. This is the Nash equilibrium in the simultaneous move game. Alternately, suppose firm F'' chooses knowing q'. Its optimal choice is the same reaction function, $q''= (A-m)/2B - q'/2$. Knowing that firm F'' will make this choice, firm F' will maximize $\pi' = (A - B(q' + (A-m)/2B - q'/2) - m)q'$ at $q' = (A-m)/2B$. Then, $q'' = (A-m)/4B$. The profit of F' is (A-m)2/8B and the profit of firm F'' is (A-m)2/16B. Then, firm F' has a higher payoff and firm F'' has a lower payoff in the sequential move case. .

4. Consider a first-price sealed bid auction of a single object with two bidders j = 1,2 and no reservation price. Bidder 1's $v_1 = 2$, and bidder 2's valuation is $v_2 = 5$. Both v_1 and v_2 are <u>known</u> to <u>both</u> bidders. Bids must be in whole dollar amounts. In the event of a tie, the object is awarded by a flip of a fair coin. Is there a Nash equilibrium? What is it? Is it unique? Is it efficient?

The complete payoff matrix is given in the following strategic game:

Player 1 \ Player 2	0	1	2	3	4	5
0	(1,5/2)	(0,4)	(0,3)	(0,2)	(0,1)	(0,0)
1	(1,0)	(1/2,2)	(0,3)	(0,2)	(0,1)	(0,0)
2	(0,0)	(0,0)	(0,3/2)	(0,2)	(0,1)	(0,0)

Bid 1 by player 1 and bid 3 by player 2 are one Nash equilibrium. However, bids 0 and 2 for player 1 are weakly dominated by bid 1, and If player 1 bids 1, then the dominant response for player 2 is to bid 2. This is another Nash equilibrium. Since player 1 has a weakly dominant strategy and player 2 has a unique best response to it, there is no need to look for mixed strategies. Both Nash equilibria are efficient, assigning the object to the higher value bidder.

5. Suppose an exchange economy has J consumers and 2 goods. Consumer j has a utility function of the form $u_j = z_j + t_j{\cdot}v_j$, where z_j is consumption of the first commodity, which is divisible, with each consumer having an initial endowment of one unit, and t_j is consumption of the second commodity, which can be consumed only in integer units, with consumer 1 having an endowment of one unit and all others having an endowment of zero. The v_j are values of the indivisible good which are drawn independently for each consumer from the uniform distribution on [0,1]. Each consumer knows her own v_j, but knows only that the v's of all others are drawn from the uniform distribution..Suppose an auction mechanism is used to assign the indivisible good, with consumer 1 participating as a possible buyer as well as the seller. Suppose this mechanism leads to a resource allocation that is in the core of the economy.

> (A) From the properties of the core, prove that the allocation produced by this auction must have resulted in the indivisible good going to the consumer with the highest v_j, with a payment by the buyer no less than the second highest v_j, and no greater than the highest v_j.
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> (B) Discuss the relationship of this result to the revenue equivalence theorem of auction theory.

An allocation in the core cannot be blocked by any coalition. Consumer 1 on her own is guaranteed utility 1 + v1 without trading, and every other consumer is guaranteed utility 1 without trading. Individual rationality says they will block any allocation that fails to deliver these utility levels. Then, if consumer 1 trades with another consumer j, the amount of the divisible goods going to the seller can be no less than v1 and no more than the highest vj. Further, if consumers 1 and, say 2, trade the indivisible good for z units of the divisible good, with z less than the highest value of a player not receiving the good, then this player and the seller have a blocking coalition that trades for this player's value of the good. This will necessarily happen unless the highest value player gets the indivisible good and the seller gets at least the second-highest value. Therefore, any auction that delivers an allocation that is in the core (i.e., efficient) implies that the final buyer has highest value and pays a price at least equal to the second highest value and no more than the highest value.

The revenue equivalence theorem says that all auction mechanisms that are efficient and individually rational yield the same expected revenue to the seller as an efficient second-price sealed bid auction. This result is consistent with, but sharper than, the implications of an allocation being in the core, establishing that the seller can get no more than the second value when bidder behavior is taken into account.