Econ 240A: Pre-midterm Review Questions

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The midterm exam for the course is closed book and notes and will take place on Tuesday 6 March in 247 Cory between 2:10 and 3:30 pm. Please bring one or two blue books and scratch paper. The exam has been set and will consist of four questions:

- one question asking for brief definitions of important concepts;
- two questions on basic probability theory;
- one multi-part question on statistical inference.

The rest of this handout consists of exercises organized around the themes of the course to date.

Basic Probability Theory

1. What are the three components of a probability space? Provide a brief definition of each component.

2. Briefly define the term *random variable*.

3. Define conditional probability. State Bayes' theorem.

4. Consider the "Gambler's Ruin": Suppose two gamblers A and B are playing a game against each other. Let p be a given number $(0 < p < 1)$, and suppose that on each play of the game, the probability that A wins one dollar from B is p and that the probability that gambler B wins one dollar from A is $q = 1 - p$. Suppose that the initial fortune of A is $I > 0$ dollars and the initial fortune of B is $K - I > 0$ dollars. The game continues until the fortune of one gambler is reduced to zero. What is the probability that A wins?

5. Consider three prisoners: Ada, Bertha, and Charlene. Ada knows that two of them are to be executed the next morning and the other set free, and she concludes that each of them is equally likely to be the lucky one that goes free. Let A denote the event that Ada goes free, and B and C similarly for Bertha and Charlene. Ada says to her jailer: "Because either Bertha or Charlene is certain to be executed, you will give me no

information about my own chances if you give me the name of one person, Bertha or Charlene, who is going to be executed with certainty." The jailer accepts this argument and truthfully says, "Bertha will be executed" (an event denoted by b). Ada chuckles to herself, because she thinks that she has tricked the jailer into giving her some information. Ada reasons that because she now knows that either she or Charlene will go free and, as before, she has no reason to think it is more likely to be Charlene, her chance of going free is now $\frac{1}{2}$ and not $\frac{1}{3}$ as before.

Is Ada acting consistently with the usual rules of probability in updating her probability of going free from $\frac{1}{3}$ to $\frac{1}{2}$? If so, show it. If not, show why not? What is the exact probability or range of probabilities for the event A that Ada can legitimately entertain after speaking with the jailer?

6. Consider the "Monty Hall Paradox": Monty Hall, the longtime host of the game show *Let's Make A Deal*, asks contestants to choose the prize behind one of three curtains. Behind one curtain is the "grand prize"; the other two curtains conceal relatively less desirable prizes. Assume Monty knows what is behind every curtain. Once the contestant has made a choice, Monty Hall reveals what is behind one of the two curtains that were not chosen. Having been shown one of the lesser prizes, the contestant is offered a chance to switch curtains. Should the contestant switch?

7. Do Exercises 1–5 from the handout on inequalities (available from the course homepage).

8. Prove the law of iterated expectations: For any function $g(X,Y)$ such that all the relevant moments exist, show that

$$E_{X|Y}[g(X,Y)] = E_X\{E_{Y|X}[g(X,Y)]\}.$$

9. Show that $E[Y] = E\{E[Y|X]\}$. Show that $Var[Y] = E\{Var[Y|X]\} + Var\{E[Y|X]\}$.

10. Do Exercise 8 from the handout on inequalities.

11. If $X \sim N(\mu, \sigma^2)$ prove that $\frac{X-\mu}{\sigma} \sim N(0,1)$.

12. If X_1, X_2 are independent and $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1,2$ prove that $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$.

13. If $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$, prove that $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \sim N(\mu, \frac{\sigma^2}{n})$.

14. If $Z_1, \ldots, Z_n$ are iid $N(0,1)$, prove that $\sum_{i=1}^n Z_i^2 \sim \chi_n^2$.

15. If $Z \sim N(0,1)$ and $X \sim \chi_n^2$ are independent, show that

$$\frac{Z}{\sqrt{\frac{X}{n}}} \sim t_n.$$

16. If X_1, X_2 are independent with $X_i \sim \chi^2_{\nu_i}$, show that

$$\frac{\frac{X_1}{\nu_1}}{\frac{X_2}{\nu_2}} \sim F(\nu_1, \nu_2).$$

Large-Sample Theory

1. Provide brief definitions of the following concepts: (i) convergence almost surely; (ii) convergence in probability; (iii) convergence in mean-square; (iv) convergence in distribution. State the relationships between the four modes of convergence.

2. Redo Exercise 1 from Chapter 4.

3. State and prove a law of large numbers. State and prove a central limit theorem.

4. Redo Exercises 2 and 3 from Chapter 4.

5. Prove the following theorem: *If Y_n is a sequence of random variables such that $Y_n \xrightarrow{p} c$, and if f is a function which is continuous at c, then $f(Y_n) \xrightarrow{p} f(c)$.*

6. Suppose X follows a *negative binomial* distribution with parameters p and m, i.e., X takes on values x over the whole numbers $\{0, 1, 2, \ldots\}$ with probability function

$$P[X = x] = \binom{m + x - 1}{m - 1} p^m (1 - p)^x.$$

Then $E[X] = \frac{m(1-p)}{p}$ and $Var[X] = \frac{m(1-p)}{p^2}$. Show that

$$\frac{X - E[X]}{\sqrt{Var[X]}} \xrightarrow{d} N(0, 1).$$

Statistical Inference

1. Define the following: (i) likelihood function; (ii) sample score; (iii) sample information.

2. Define the following: (i) sufficiency; (ii) minimal sufficiency.

3. Prove Lemma 1 in the handout on sufficiency (available from the course homepage).

4. Define the following: (i) admissibility; (ii) Bayes risk.

5. Suppose $X_1, \ldots, X_n$ are iid $N(\mu, 1)$. Suppose we have prior beliefs on μ represented by a $N(\nu, \tau^2)$ distribution. Derive the posterior distribution of μ.

6. Give a careful definition of a confidence interval.

7. Suppose $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$. Describe the properties of the statistics $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

8. Redo Exercise 1 from Chapter 7. Give careful derivations of test statistics, critical values, and power functions.

9. Do Exercise 3 from Chapter 7.